Moduli of metaplectic bundles on curves and Theta-sheaves

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Abstract We give a geometric interpretation of the Weil representation of the metaplectic group, placing it in the framework of the geometric Langlands program.

For a smooth projective curve X we introduce an algebraic stack $\widetilde{\operatorname{Bun}}_G$ of metaplectic bundles on X. It also has a local version $\widetilde{\operatorname{Gr}}_G$, which is a gerbe over the affine grassmanian of G. We define a categorical version of the (nonramified) Hecke algebra of the metaplectic group. This is a category $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$ of certain perverse sheaves on $\widetilde{\operatorname{Gr}}_G$, which act on $\widetilde{\operatorname{Bun}}_G$ by Hecke operators. A version of the Satake equivalence is proved describing $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$ as a tensor category. Further, we construct a perverse sheaf on $\widetilde{\operatorname{Bun}}_G$ corresponding to the Weil representation and show that it is a Hecke eigen-sheaf with respect to $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$.

1. Introduction

1.1 Historically θ-series (such as, in one variable, $\sum q^{n^2}$) have been one of the major methods of constructing automorphic forms. A representation-theoretic appoach to the theory of θ-series, as discoved by A. Weil [20] and extended by R. Howe [12], is based on the oscillator representation of the metaplectic group (cf. [19] for a recent survey). In this paper we propose a geometric interpretation of this representation (in the nonramified case) placing it in the framework of the geometric Langlands program.

Let $k = \mathbb{F}_q$ be a finite field with q odd. Set $K = k((t))$ and $\mathcal{O} = k[[t]]$. Let Ω denote the completed module of relative differentials of $\mathcal{O}$ over k. Let M be a free $\mathcal{O}$-module of rank $2n$ given with a nondegenerate symplectic form $\wedge^2 M \to \Omega$. It is known that the continuous $\mathrm{H}^2(\mathbb{S}\mathrm{p}(M)(K), \{\pm 1\}) \widetilde{\to} \mathbb{Z}/2\mathbb{Z}$ ([16], 10.4). As $\mathbb{S}\mathrm{p}(M)(K)$ is a perfect group, the corresponding metaplectic extension

$$1 \to \{\pm 1\} \xrightarrow{i} \widehat{\mathbb{S}\mathrm{p}}(M)(K) \to \mathbb{S}\mathrm{p}(M)(K) \to 1 \qquad (1)$$

is unique up to unique isomorphism. It can be constructed in two essentially different ways.

Recall the classical construction of A. Weil ([20]). The Heisenberg group is $H(M) = M \oplus \Omega$ with operation

$$(m_1, \omega_1)(m_2, \omega_2) = (m_1 + m_2,\, \omega_1 + \omega_2 + \frac{1}{2}\langle m_1, m_2\rangle)$$

Fix a prime ℓ that does not divide q. Let $\psi : k \to \bar{\mathbb{Q}}_\ell^*$ be a nontrivial additive character. Let $\chi : \Omega(K) \to \bar{\mathbb{Q}}_\ell$ be given by $\chi(\omega) = \psi(\operatorname{Res}\omega)$. By the Stone and Von Neumann theorem ([18]), there is a unique (up to isomorphism) smooth irreducible representation $(\rho, \mathcal{S}_\psi)$ of $H(M)(K)$

over $\bar{\mathbb{Q}}_\ell$ with central character χ. The group $\mathbb{S}\mathrm{p}(M)$ acts on $H(M)$ by group automorphisms $(m,\omega)\xrightarrow{g}(gm,\omega)$ This gives rise to the group

$$\begin{aligned}\widetilde{\mathbb{S}\mathrm{p}}(M)(K)=\{(g,M[g])\mid g\in\mathbb{S}\mathrm{p}(M)(K), M[g]\in\operatorname{Aut}\mathcal{S}_\psi\\ \rho(gm,\omega)\circ M[g]=M[g]\circ\rho(m,\omega)\ \text{ for } (m,\omega)\in H(M)(K)\}\end{aligned}$$

The group $\widetilde{\mathbb{S}\mathrm{p}}(M)(K)$ is an extension of $\mathbb{S}\mathrm{p}(M)(K)$ by $\bar{\mathbb{Q}}_\ell^*$. Its commutator subgroup is an extension of $\mathbb{S}\mathrm{p}(M)(K)$ by $\{\pm 1\}\hookrightarrow\bar{\mathbb{Q}}_\ell^*$, uniquely isomorphic to (1).

Another way is via Kac-Moody groups. Namely, view $\mathbb{S}\mathrm{p}(M)(K)$ as an ind-scheme over k. Let

$$1\to\mathbb{G}_m\to\overline{\mathbb{S}\mathrm{p}}(M)(K)\to\mathbb{S}\mathrm{p}(M)(K)\to 1 \tag{2}$$

denote the canonical extension, here $\overline{\mathbb{S}\mathrm{p}}(M)(K)$ is an ind-scheme over k (cf. [10]). Passing to k-points we get an extension of abstract groups $1\to k^*\to\overline{\mathbb{S}\mathrm{p}}(M)(K)\to\mathbb{S}\mathrm{p}(M)(K)\to 1$. Then (1) is the push-forward of this extension under $k^*\to k^*/(k^*)^2$.

The second construction underlies one of our main results, the tannakian description of the Langlands dual to the metaplectic group. Namely, the canonical splitting of (2) over $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$ yields a splitting of (1) over $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$. Consider the Hecke algebra

$$\begin{aligned}\mathcal{H}=\{f:\mathbb{S}\mathrm{p}(M)(\mathcal{O})\backslash\widehat{\mathbb{S}\mathrm{p}}(M)(K)/\mathbb{S}\mathrm{p}(M)(\mathcal{O})\to\bar{\mathbb{Q}}_\ell\mid f(i(-1)g)=-f(g),\ \ g\in\widehat{\mathbb{S}\mathrm{p}}(M)(K);\\ f\text{ is of compact support}\}\end{aligned}$$

The product is convolution, defined using the Haar measure on $\widehat{\mathbb{S}\mathrm{p}}(M)(K)$ for which the inverse image of $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$ has volume 1.

Set $G=\mathbb{S}\mathrm{p}(M)$. Let $\check{G}$ denote $\mathbb{S}\mathrm{p}_{2n}$ viewed as an algebraic group over $\bar{\mathbb{Q}}_\ell$. Let $\mathrm{Rep}(\check{G})$ denote the category of finite-dimensional representations of $\check{G}$. Write $K(\mathrm{Rep}(\check{G}))$ for the Grothendieck ring of $\mathrm{Rep}(\check{G})$ over $\bar{\mathbb{Q}}_\ell$. There is a canonical isomorphism of $\bar{\mathbb{Q}}_\ell$-algebras

$$\mathcal{H}\,\widetilde{\to}\,K(\mathrm{Rep}(\check{G}))$$

Actually, a categorical version of this isomorphism is proved. Consider the affine grassmanian $\mathrm{Gr}_G=G(K)/G(\mathcal{O})$, viewed as an ind-scheme over k. Let W denote the nontrivial ℓ-adic local system of rank one on $\mathbb{G}_m$ corresponding to the covering $\mathbb{G}_m\to\mathbb{G}_m$, $x\mapsto x^2$. Denote by $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ the category of $G(\mathcal{O})$-equivariant perverse sheaves on $\overline{G}(K)/G(\mathcal{O})$, which are also $(\mathbb{G}_m,W)$-equivariant. Here $\widetilde{\mathrm{Gr}}_G$ denotes the stack quotient of $\overline{G}(K)$ by $\mathbb{G}_m$ with respect to the action $g\xrightarrow{x}x^2g$, $x\in\mathbb{G}_m, g\in\overline{G}(K)$. Actually, $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is a full subcategory of the category of perverse sheaves on $\widetilde{\mathrm{Gr}}_G$.

Assuming for simplicity k algebraically closed, we equip $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ with the structure of a rigid tensor category. We establish a canonical equivalence of tensor categories

$$\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)\,\widetilde{\to}\,\mathrm{Rep}(\check{G})$$

1.2 In the global setting let X be a smooth projective curve over k. Let G denote the sheaf of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ (now Ω is the canonical line bundle on X) preserving the symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$. The stack Bun_G of G-bundles (=G-torsors) on X classifies vector bundles M of rank $2n$ on X, given with a nondegenerate symplectic form $\wedge^2 M \to \Omega$. We introduce an algebraic stack $\widetilde{\mathrm{Bun}}_G$ of metaplectic bundles on X. The stack $\widetilde{\mathrm{Gr}}_G$ is a local version of $\widetilde{\mathrm{Bun}}_G$. The category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ acts on $\mathrm{D}(\widetilde{\mathrm{Bun}}_G)$ by Hecke operators.

We construct a perverse sheaf Aut on $\widetilde{\mathrm{Bun}}_G$, a geometric analog of the Weil representation. We calculate the fibres of Aut and its constant terms for maximal parabolic subgroups of G. Finally, we argue that Aut is a Hecke eigensheaf on $\widetilde{\mathrm{Bun}}_G$ with eigenvalue

$$\mathrm{St} = \mathrm{R}\Gamma(\mathbb{P}^{2n-1}, \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes 2n-1}$$

viewed as a constant complex on X. Note that St is equipped with an action of SL_2 of Arthur, the corresponding representation of SL_2 is irreducible of dimension $2n$ and admits a unique, up to a multiple, symplectic form. One may imagine that Aut corresponds to a group homomorphism $\pi_1(X) \times \mathrm{SL}_2 \to \check{G}$ trivial on $\pi_1(X)$. This agrees with Arthur's conjectures.

2. Weil representation and motivations

2.1 Let X be a smooth projective absolutely irreducible curve over $k = \mathbb{F}_q$, $F = \mathbb{F}_q(X)$, $\mathbb{A}$ be the adeles rings of F, $\mathcal{O} \subset \mathbb{A}$ be the entire adeles. Assume that q is odd. Fix a prime ℓ that does not divide q. Let Ω denote the canonical line bundle on X.

Let M be a $2n$-dimensional vector space over F with symplectic form $\wedge^2 M \to \Omega_F$, where Ω_F is the generic fibre of Ω. The Heisenberg group $H(M) = M \oplus \Omega_F$ with operation

$$(m_1, \omega_1)(m_2, \omega_2) = (m_1 + m_2,\, \omega_1 + \omega_2 + \frac{1}{2}\langle m_1, m_2\rangle)$$

is algebraic over F. Fix a nontrivial additive character $\psi : \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell^*$. Then $H(M)(\mathbb{A}) = M(\mathbb{A}) \oplus \Omega(\mathbb{A})$ admits a canonical central character $\chi : \Omega(\mathbb{A})/\Omega(F) \to \bar{\mathbb{Q}}_\ell^*$ given by

$$\chi(\omega) = \psi(\sum_{x\in X} \mathrm{tr}_{k(x)/k} \operatorname{Res} \omega_x)$$

The Stone and Von Neumann theorem ([18]) says that there is a unique (up to isomorphism) smooth irreducible representation $(\rho, \mathcal{S}_\psi)$ of $H(M)(\mathbb{A})$ over $\bar{\mathbb{Q}}_\ell$ with central character χ. The group $\mathbb{S}\mathrm{p}(M)$ acts on $H(M)$ by group automorphisms $(m, \omega) \xrightarrow{g} (gm, \omega)$. This defines the global metaplectic group[1]

$$\begin{aligned}\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A}) = \{(g, M[g]) \mid g \in \mathbb{S}\mathrm{p}(M)(\mathbb{A}), M[g] \in \operatorname{Aut} \mathcal{S}_\psi \\ \rho(gm, \omega) \circ M[g] = M[g] \circ \rho(m, \omega) \;\text{ for } (m, \omega) \in H(M)(\mathbb{A})\}\end{aligned}$$

[1]the notation $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$ is ambiguous, these are not $\mathbb{A}$-points of an algebraic group.

included into an exact sequence

$$1 \to \bar{\mathbb{Q}}_\ell^* \to \widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A}) \to \mathbb{S}\mathrm{p}(M)(\mathbb{A}) \to 1 \tag{3}$$

The representation of $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$ on S_ψ is called the Weil (or oscillator) representation ([20]).

For a subgroup $K \subset \mathbb{S}\mathrm{p}(M)(\mathbb{A})$ write $\tilde{K}$ for the preimage of K in $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$. Since χ is trivial on Ω_F, one may talk about $H(M)$-invariant functionals on S_ψ, they are called theta-functionals. The space of theta-functionals is 1-dimensional and preserved by $\widetilde{\mathbb{S}\mathrm{p}}(M)(F)$, so the action of $\widetilde{\mathbb{S}\mathrm{p}}(M)(F)$ on this space defines a splitting of (3) over $\mathbb{S}\mathrm{p}(M)(F)$.

View

$$\mathrm{Funct}(\mathbb{S}\mathrm{p}(M)(F)\backslash\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})) = \{f : \mathbb{S}\mathrm{p}(M)(F)\backslash\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})) \to \bar{\mathbb{Q}}_\ell\}$$

as a representation of $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$ by right translations. A theta-functional $\Theta : S_\psi \to \bar{\mathbb{Q}}_\ell$ defines a morphism of $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$-modules

$$\mathcal{S}_\psi \to \mathrm{Funct}(\mathbb{S}\mathrm{p}(M)(F)\backslash\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})) \tag{4}$$

sending ϕ to θ_ϕ given by $\theta_\phi(g) = \Theta(g\phi)$ for $g \in \widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})$.

Now assume that M is actually a rank $2n$ vector bundle on X with symplectic form $\wedge^2 M \to \Omega$. Then we get the subgroups $\mathbb{S}\mathrm{p}(M)(\mathcal{O}) \subset \mathbb{S}\mathrm{p}(M)(\mathbb{A})$ and $M(\mathcal{O}) \oplus \Omega(\mathcal{O}) \subset H(M)(\mathbb{A})$. Moreover, the space of $M(\mathcal{O}) \oplus \Omega(\mathcal{O})$-invariants in $\mathcal{S}_\psi$ is 1-dimensional and preserved by $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathcal{O})$. The action of $\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathcal{O})$ on this space yields a splitting of (3) over $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$. If $\phi_0 \in \mathcal{S}_\psi$ is a nonzero $M(\mathcal{O}) \oplus \Omega(\mathcal{O})$-invariant vector then its image under (4) is the classical theta-function

$$f_0 : \mathbb{S}\mathrm{p}(M)(F)\backslash\widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A}))/\,\mathbb{S}\mathrm{p}(M)(\mathcal{O}) \to \bar{\mathbb{Q}}_\ell$$

that we are going to geometrize.

Let G denote the sheaf of automorphisms of M preserving the form $\wedge^2 M \to \Omega$. This is a sheaf of groups (in flat topology) on X locally in Zarisky topology isomorphic to $\mathbb{S}\mathrm{p}_{2n}$.

2.2 Assume $M = V \oplus (V^* \otimes \Omega)$ is a direct sum of lagrangian subbundles, the form being given by the canonical pairing $\langle .,. \rangle$ between V and V^*. Let

$$\chi_V : V(\mathbb{A}) \oplus \Omega(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^*$$

denote the character $\chi_V(v,\omega) = \chi(\omega)$.

We have the subgroup $V(\mathbb{A}) \subset H(M)(\mathbb{A})$. The space of $V(\mathbb{A})$-invariant functionals on $\mathcal{S}_\psi$ is 1-dimensional. A choice of such functional identifies $\mathcal{S}_\psi$ with the induced representation of $(V(\mathbb{A}) \oplus \Omega(\mathbb{A}), \chi_V)$ to $H(M)(\mathbb{A})$. The latter identifies with the Schwarz space $\mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$ of locally constant compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $V^* \otimes \Omega(\mathbb{A})$, the corresponding functional on $\mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$ becomes the evaluation at zero $ev : \mathcal{S}(V^* \otimes \Omega(\mathbb{A})) \to \bar{\mathbb{Q}}_\ell$. This is the Schrödinger model of $\mathcal{S}_\psi$.

Write $g \in \mathbb{S}\mathrm{p}(M)(\mathbb{A})$ as a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5}$$

with $a \in \mathrm{End}(V)(\mathbb{A}), b \in \mathrm{Hom}(V^* \otimes \Omega, V)(\mathbb{A}), d \in \mathrm{End}(V^*)(\mathbb{A}), c \in \mathrm{Hom}(V, V^* \otimes \Omega)(\mathbb{A})$. Write a^* for the transpose operator to a.

The defined up to a scalar automorphism $M[g]$ of $\mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$ is described as follows.

- For $a \in \mathrm{GL}(V)(\mathbb{A})$ we have $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} \in \mathbb{S}\mathrm{p}(M)(\mathbb{A})$. Besides, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathbb{S}\mathrm{p}(M)(\mathbb{A})$ if and only if $b \in (V \otimes V \otimes \Omega^{-1})(\mathbb{A})$ is symmetric. For g given by (5) with $c = 0$ we have

$$(M[g]f)(v^*) = \chi(\frac{1}{2}\langle a^*v^*, b^*v^*\rangle) f(a^*v^*), \quad v^* \in V^* \otimes \Omega(\mathbb{A}) \tag{6}$$

- if $b : V^* \otimes \Omega(\mathbb{A}) \widetilde{\to} V(\mathbb{A})$ then $g = \begin{pmatrix} 0 & b \\ -b^{*-1} & 0 \end{pmatrix} \in \mathbb{S}\mathrm{p}(M)(\mathbb{A})$ and

$$(M[g]f)(v^*) = \int_{V(\mathbb{A})} \chi(\langle v, v^*\rangle) f(b^{-1}v) dv, \quad v^* \in V^* \otimes \Omega(\mathbb{A}) \tag{7}$$

  for any Haar measure dv on $V(\mathbb{A})$.

Let $P \subset G$ denote the Siegel parabolic subgroup preserving V. The subgroup $\tilde{P}(\mathbb{A})$ preserves ev up to a multiple, so defining a splitting of (3) over $P(\mathbb{A})$. This splitting coincides with the one given by (6).

Let $\phi_0 \in \mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$ denote the characteristic function of $V^* \otimes \Omega(\mathcal{O})$. Using (6) and (7) one shows that ϕ_0 generates the space of $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$-invariants in $\mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$. In this model of $\mathcal{S}_\psi$ the theta functional $\Theta : \mathcal{S}(V^* \otimes \Omega(\mathbb{A})) \to \bar{\mathbb{Q}}_\ell$ is given by

$$\Theta(\phi) = \sum_{v^* \in V^* \otimes \Omega(F)} \phi(v^*) \quad \text{for } \phi \in \mathcal{S}(V^* \otimes \Omega(\mathbb{A}))$$

Let f_0 denote the image of ϕ_0 under the corresponding map (4). Let us calculate the composition

$$P(F)\backslash P(\mathbb{A})/P(\mathcal{O}) \to \mathbb{S}\mathrm{p}(M)(F)\backslash \widetilde{\mathbb{S}\mathrm{p}}(M)(\mathbb{A})/\,\mathbb{S}\mathrm{p}(M)(\mathcal{O}) \stackrel{f_0}{\to} \bar{\mathbb{Q}}_\ell$$

denoted f_P. We used the fact that the splittings of (3) over $P(\mathbb{A})$ and $\mathbb{S}\mathrm{p}(M)(\mathcal{O})$ are compatible over $P(\mathcal{O})$.

Denote by Bun_n the k-stack of rank n vector bundles on X. The set $\mathrm{GL}(V)(\mathbb{A})/\mathrm{GL}(V)(\mathcal{O})$ naturally identifies with the isomorphism classes of pairs (L, α), where $L \in \mathrm{Bun}_n(k)$ and $\alpha : L(F) \widetilde{\to} V(F)$. Here $L(F)$ is the generic fibre of L.

Let $a \in \mathrm{GL}(V)(\mathbb{A})$ and (L, α) be the pair attached to $a\,\mathrm{GL}(V)(\mathcal{O})$. Then

$$\{v^* \in V^* \otimes \Omega(F) \mid a^*v^* \in V^* \otimes \Omega(\mathcal{O})\} \stackrel{\alpha^*}{\to} \mathrm{Hom}(L, \Omega) \tag{8}$$

is an isomorphism.

The group P fits into an exact sequence $1 \to (\mathrm{Sym}^2 V) \otimes \Omega^{-1} \to P \to \mathrm{GL}(V) \to 1$ of algebraic groups over X. For $g \in P(\mathbb{A})$ we get

$$f_P(g) = \Theta(g\phi_0) = \sum_{v^* \in V^* \otimes \Omega(F)} (g\phi_0)(v^*) = \sum_{v^* \in V^* \otimes \Omega(F)} \chi(\frac{1}{2}\langle a^* v^*, b^* v^*\rangle)\phi_0(a^* v^*) =$$

$$\sum_{s \in \mathrm{Hom}(L,\Omega)} \chi(\frac{1}{2}\langle s, ab^* s\rangle)$$

in view of (8).

Let Bun_P be the k-stack of P-bundles on X. Its Y-points for a scheme Y is the category of $(Y \times X) \times_X P$-torsors over $Y \times X$. Then Bun_P classifies pairs $L \in \mathrm{Bun}_n$ together with an exact sequence on X

$$0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0 \tag{9}$$

(More generally, for a semidirect product of group schemes $1 \to U \to P \to M \to 1$ providing a P-torsor $\mathcal{F}_P$ is equivalent to providing a M-torsor $\mathcal{F}_M$ and a $U_{\mathcal{F}_M}$-torsor of isomorphisms $\mathrm{Isom}(\mathcal{F}_P, \mathcal{F}_M \times_M P)$ inducing a given one on the corresponding M-torsors).

In view of the bijection $P(F)\backslash P(\mathbb{A})/P(\mathcal{O}) \widetilde{\to} \mathrm{Bun}_P(k)$, the function f_P on $\mathrm{Bun}_P(k)$ is described as follows. Let a P-torsor $\mathcal{F}_P \in \mathrm{Bun}_P(k)$ be given by $L \in \mathrm{Bun}_n(k)$ together with (9). Consider the map $q^{\mathcal{F}_P} : \mathrm{Hom}(L, \Omega) \to k$ sending $s \in \mathrm{Hom}(L, \Omega)$ to the pairing of

$$s \otimes s \in \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^{\otimes 2})$$

with the exact sequence (9). Then

$$f_P(\mathcal{F}_P) = \sum_{s \in \mathrm{Hom}(L,\Omega)} \psi(q^{\mathcal{F}_P}(s))$$

The function $f_P : \mathrm{Bun}_P(k) \to \bar{\mathbb{Q}}_\ell$ is the trace of Frobenius of the following ℓ-adic complex $S_{P,\psi}$ on Bun_P.

Let $p : \mathcal{X} \to \mathrm{Bun}_P$ be the stack over Bun_P with fibre $\mathrm{Hom}(L, \Omega)$. Let $q : \mathcal{X} \to \mathbb{A}^1$ be the map sending $s \in \mathrm{Hom}(L, \Omega)$ to the pairing of (9) with

$$s \otimes s \in \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^{\otimes 2})$$

The geometric analog of f_P is the complex $S_{P,\psi} = p_! q^* \mathcal{L}_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{X}}$ on Bun_P, here $\dim \mathcal{X}$ denotes the dimension of the corresponding connected component of $\mathcal{X}$.

3. Main results

3.1 Notation From now on k denotes an algebraically closed field of characteristic $p > 2$, all the schemes (or stacks) we consider are defined over k.

Let X be a smooth projective connected curve. Write Ω for the canonical line bundle on X. Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\mathrm{D}(S)$ for the bounded derived category of ℓ-adic étale sheaves on S, and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves (the middle perversity function is always taken in absolut sense over $\operatorname{Spec} k$).

Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$. Fix a square root $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ of the sheaf $\bar{\mathbb{Q}}_\ell(1)$ on $\operatorname{Spec} \mathbb{F}_q$. Isomorphism classes of such correspond to square roots of q in $\bar{\mathbb{Q}}_\ell$. Fix an inclusion of fields $\mathbb{F}_q \hookrightarrow k$.

If $V \to S$ and $V^* \to S$ are dual rank n vector bundles over a stack S, we normalize the Fourier trasform $\mathrm{Four}_\psi : \mathrm{D}(V) \to \mathrm{D}(V^*)$ by $\mathrm{Four}_\psi(K) = (p_{V^*})_!(\xi^*\mathcal{L}_\psi \otimes p_V^* K)[n](\frac{n}{2})$, where p_V, p_{V^*} are the projections, and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

A G-torsor on a scheme S is also referred to as a G-bundle on S. Write Vect^ϵ for the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, our conventions about this category are those of ([7]). Write $\mathrm{Vect} \subset \mathrm{Vect}^\epsilon$ for its even component, i.e., the tensor category of vector spaces.

3.1.1 The sheaf (in flat topology) on the category of k-schemes represented by $\mu_2 := \operatorname{Ker}(x \mapsto x^2 : \mathbb{G}_m \to \mathbb{G}_m)$ is the constant sheaf $\{\pm 1\}$.

For a scheme S and a line bundle $\mathcal{A}$ on S denote by $\tilde{S}$ the following μ_2-gerbe over S. For an S-scheme S', the category of S'-points of $\tilde{S}$ is the category of pairs $(\mathcal{B},\ \mathcal{B}^2 \widetilde{\to} \mathcal{A}\mid_{S'})$, where $\mathcal{B}$ is a line bundle on S'. Note that $\tilde{S} \to S$ is étale.

If $\tilde{S} \to S$ admits a section given by invertible $\mathcal{O}_S$-module $\mathcal{B}_0$ together with $\mathcal{B}_0^2 \widetilde{\to} \mathcal{A}$ then the gerbe is trivial, that is, $\tilde{S} \widetilde{\to} B(\mu_2/S)$ over S. In this case we get the S_2-covering $\mathrm{Cov}(\tilde{S}) \to \tilde{S}$, whose fibre consists of isomorphisms $\mathcal{B} \widetilde{\to} \mathcal{B}_0$ whose square is the given one $\mathcal{B}^2 \widetilde{\to} \mathcal{A}$. This covering is locally trivial in étale topology, but not trivial even for $S = \operatorname{Spec} k$. Actually $S = \mathrm{Cov}(\tilde{S})$.

3.1.2 If in addition $\mathcal{A}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on S purely of degree zero, then by definition $\tilde{S}$ classifies a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $\mathcal{B}$ purely of degree zero, given with a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\mathcal{B}^2 \widetilde{\to} \mathcal{A}$. If $\mathcal{B}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on S *of pure degree* (that is, placed in one degree only over each connected component) then a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\mathcal{B}^2 \widetilde{\to} \mathcal{A}$ yields a (uniquely defined) section of $\tilde{S}$.

3.2 Let Bun_n be the stack of rank n vector bundles on X. Let G denote the sheaf of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ preserving the symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$. So, G is a sheaf of groups in flat topology on the category of X-schemes.

The stack Bun_G of G-bundles on X classifies $M \in \mathrm{Bun}_{2n}$ together with a symplectic form $\wedge^2 M \to \Omega$. This is a smooth algebraic stack locally of finite type over k. Since G is simply-connected, Bun_G is irreducible ([6], 2.2.1). Let $d_G = \dim \mathrm{Bun}_G = (g-1)\dim \mathfrak{sp}_{2n}$. To express the dependence on n we write G_n, Bun_{G_n}, d_{G_n} and so on.

Denote by $\mathcal{A}$ the line bundle on Bun_G whose fibre at M is $\det \mathrm{R}\Gamma(X, M)$ (cf. [7]). As $\chi(M) = 0$, we view $\mathcal{A}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle placed in degree zero. It yields a μ_2-gerbe

$$\mathfrak{r} : \widetilde{\mathrm{Bun}}_G \to \mathrm{Bun}_G \tag{10}$$

So, S-points of $\widetilde{\mathrm{Bun}}_G$ is the category: a line bundle $\mathcal{B}$ on S, a vector bundle M on $S \times X$ of rank $2n$ with symplectic form $\wedge^2 M \to \Omega_{S\times X/S}$, and an isomorphism of $\mathcal{O}_S$-modules $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$.

The idea of using the determinant of cohomology was communicated to me by G. Laumon and goes back to P. Deligne [8].

Let ${}_i\mathrm{Bun}_G \hookrightarrow \mathrm{Bun}_G$ be the locally closed substack given by $\dim \mathrm{H}^0(X, M) = i$. Let ${}_i\widetilde{\mathrm{Bun}}_G$ denote the preimage of ${}_i\mathrm{Bun}_G$ under $\mathfrak{r}$.

Lemma 1. *Each stratum ${}_i\mathrm{Bun}_G$ of Bun_G is nonempty.*

Proof For $n = 1$ take $M = \mathcal{A}(D) \oplus (\mathcal{A}^* \otimes \Omega(-D))$, where D is an effective divisor of degree i on X, and $\mathcal{A}$ is a line bundle on X of degree $g-1$ such that $\mathrm{H}^0(X, \mathcal{A}) = \mathrm{H}^1(X, \mathcal{A}) = 0$. Such $\mathcal{A}$ exist, because $\dim X^{(g-1)} = g-1$, and the dimension of the Picard scheme of X is g. Then $\dim \mathrm{H}^0(X, M) = i$.

For any n construct $M \in {}_i\mathrm{Bun}_G$ as $M = M_1 \oplus \ldots \oplus M_n$ with $M_j \in {}_{i_j}\mathrm{Bun}_{G_1}$ for some $i_1 + \ldots + i_n = i$. $\square$

We have a line bundle ${}_i\mathcal{B}$ on ${}_i\mathrm{Bun}_G$ whose fibre at $M \in \mathrm{Bun}_G$ is $\det \mathrm{H}^0(X, M)$. View it as a $\mathbb{Z}/2\mathbb{Z}$-graded placed in degree $\dim \mathrm{H}^0(X, M)$ modulo 2. Then for each i we get a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism ${}_i\mathcal{B}^2 \widetilde{\to} \mathcal{A} \mid_{{}_i\mathrm{Bun}_G}$. By 3.1.2, the gerbe ${}_i\widetilde{\mathrm{Bun}}_G \to {}_i\mathrm{Bun}_G$ is trivial. So, we have the two-sheeted covering

$$ {}_i\rho : \mathrm{Cov}({}_i\widetilde{\mathrm{Bun}}_G) \to {}_i\widetilde{\mathrm{Bun}}_G $$

The line bundles ${}_i\mathcal{B}$ (viewed as ungraded) do not glue into a line bundle over Bun_G (the gerbe $\mathfrak{r}$ is nontrivial, because $\mathcal{A}$ is a generator of the Picard group $\mathrm{Pic}(\mathrm{Bun}_G) \widetilde{\to} \mathbb{Z}$ by [10]).

Definition 1. For each i define a local system ${}_i\mathrm{Aut}$ on ${}_i\widetilde{\mathrm{Bun}}_G$ by

$$ {}_i\mathrm{Aut} = \mathrm{Hom}_{S_2}(\mathrm{sign}, {}_i\rho_! \bar{\mathbb{Q}}_\ell) $$

Let $\mathrm{Aut}_g \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$ (resp., $\mathrm{Aut}_s \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$) denote the Goresky-MacPherson extension of ${}_0\mathrm{Aut} \otimes \bar{\mathbb{Q}}_\ell[d_G](\frac{d_G}{2})$ (resp., of ${}_1\mathrm{Aut} \otimes \bar{\mathbb{Q}}_\ell[d_G - 1](\frac{d_G - 1}{2})$) under ${}_i\widetilde{\mathrm{Bun}}_G \hookrightarrow \widetilde{\mathrm{Bun}}_G$. [2] Set

$$ \mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s $$

By construction, $\mathbb{D}(\mathrm{Aut}) \widetilde{\to} \mathrm{Aut}$ canonically.

Here is our main result.

Theorem 1. *For each i the $*$-restriction $\mathrm{Aut} \mid_{{}_i\widetilde{\mathrm{Bun}}_G}$ identifies with*

$$ \mathrm{Aut} \mid_{{}_i\widetilde{\mathrm{Bun}}_G} \widetilde{\to} {}_i\mathrm{Aut} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_G - i}, $$

(once $\sqrt{-1} \in k$ is fixed, the corresponding isomorphism is well-defined up to a sign). The $*$-restriction of Aut_g (resp., of Aut_s) to ${}_i\widetilde{\mathrm{Bun}}_G$ vanishes for i odd (resp., even).

[2] Here 'g' stands for generic and 's' for special. We postpone to Proposition 7 the proof of the fact that ${}_1\mathrm{Aut}$ is a shifted perverse sheaf on ${}_1\widetilde{\mathrm{Bun}}_G$

Remark 1. Classicaly, for two symplectic spaces W, W' there is a natural map $\widetilde{\mathbb{S}\mathrm{p}}(W)\times\widetilde{\mathbb{S}\mathrm{p}}(W')\to\widetilde{\mathbb{S}\mathrm{p}}(W\oplus W')$, and the restriction of the metaplectic representation under this map is the tensor product of metaplectic representations of the factors ([19], Remark 2.7).

In geometric setting we have a map $s_{n,m}:\mathrm{Bun}_{G_n}\times\mathrm{Bun}_{G_m}\to\mathrm{Bun}_{G_{n+m}}$ sending M, M' to $M\oplus M'$. It extends to a map

$$\tilde s_{n,m}:\widetilde{\mathrm{Bun}}_{G_n}\times\widetilde{\mathrm{Bun}}_{G_m}\to\widetilde{\mathrm{Bun}}_{G_{n+m}}$$

sending $(M,\mathcal{B},\mathcal{B}^2\widetilde{\to}\det\mathrm{R}\Gamma(X,M))$ and $(M',\mathcal{B}',\mathcal{B}'^2\widetilde{\to}\det\mathrm{R}\Gamma(X,M'))$ to

$$(M\oplus M',\mathcal{B}\otimes\mathcal{B}',\mathcal{B}^2\otimes\mathcal{B}'^2\widetilde{\to}\det\mathrm{R}\Gamma(X,M)\otimes\det\mathrm{R}\Gamma(X,M')\widetilde{\to}\det\mathrm{R}\Gamma(X,M\oplus M'))$$

The restriction yields a map $s_{n,m}: {}_i\mathrm{Bun}_{G_n}\times{}_j\mathrm{Bun}_{G_m}\to{}_{i+j}\mathrm{Bun}_{G_{n+m}}$ and we get canonically $s^*_{n,m}({}_{i+j}\mathcal{B})\widetilde{\to}{}_i\mathcal{B}\boxtimes{}_j\mathcal{B}$. For any i,j this yields an isomorphism

$$\tilde s^*_{n,m}({}_{i+j}\mathrm{Aut})\widetilde{\to}{}_i\mathrm{Aut}\boxtimes{}_j\mathrm{Aut}$$

of local systems on ${}_i\widetilde{\mathrm{Bun}}_{G_n}\times{}_j\widetilde{\mathrm{Bun}}_{G_m}$. Thus,

$$\tilde s^*_{n,m}\mathrm{Aut}_g\otimes\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_{G_n}+d_{G_m}-d_{G_{n+m}}}\widetilde{\to}(\mathrm{Aut}_g\boxtimes\mathrm{Aut}_g)\oplus(\mathrm{Aut}_s\boxtimes\mathrm{Aut}_s)$$

and

$$\tilde s^*_{n,m}\mathrm{Aut}_s\otimes\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_{G_n}+d_{G_m}-d_{G_{n+m}}}\widetilde{\to}(\mathrm{Aut}_g\boxtimes\mathrm{Aut}_s)\oplus(\mathrm{Aut}_s\boxtimes\mathrm{Aut}_g)$$

in the completed Grothendieck group $K(\widetilde{\mathrm{Bun}}_{G_n}\times\widetilde{\mathrm{Bun}}_{G_m})$ (the completion is with respect to the filtration given by the codimension of support).

3.3 For $1\le k\le n$ denote by Bun_{P_k} the stack classifying $M\in\mathrm{Bun}_G$ together with an isotropic subbundle $L_1\subset M$ of rank k. We write $L_{-1}\subset M$ for the orthogonal complement of L_1, so a point of Bun_{P_k} gives rise to a flag $(L_1\subset L_{-1}\subset M)$, and $L_{-1}/L_1\in\mathrm{Bun}_{G_{n-k}}$ naturally.

Write $\nu_k:\mathrm{Bun}_{P_k}\to\mathrm{Bun}_G$ for the projection. Define the map

$$\tilde\nu_k:\widetilde{\mathrm{Bun}}_{G_{n-k}}\times_{\mathrm{Bun}_{G_{n-k}}}\mathrm{Bun}_{P_k}\to\widetilde{\mathrm{Bun}}_G$$

as follows. An S-point of the source is given by $(L_1\subset L_{-1}\subset M)\in\mathrm{Bun}_{P_k}(S)$ together with a ($\mathbb{Z}/2\mathbb{Z}$-graded of pure degree zero) invertible $\mathcal{O}_S$-module $\mathcal{B}$ and $\mathcal{B}^2\widetilde{\to}\det\mathrm{R}\Gamma(X,L_{-1}/L_1)$. We have a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded lines

$$\det\mathrm{R}\Gamma(X,L_1)\otimes\det\mathrm{R}\Gamma(X,L_{-1}/L_1)\otimes\det\mathrm{R}\Gamma(X,L_1^*\otimes\Omega)\widetilde{\to}\det\mathrm{R}\Gamma(X,M)\tag{11}$$

The map $\tilde\nu_k$ sends this point to $M\in\mathrm{Bun}_G$ together with an invertible $\mathcal{O}_S$-module $\mathcal{B}'=\mathcal{B}\otimes\det\mathrm{R}\Gamma(X,L_1)$ and $\mathcal{B}'^2\widetilde{\to}\det\mathrm{R}\Gamma(X,M)$ given by (11). Since $\mathcal{B}'$ is of pure degree as $\mathbb{Z}/2\mathbb{Z}$-graded, the map is well-defined by 3.1.2.

Let Bun_{Q_k} be the stack of collections: an exact sequence $0\to L_1\to L_{-1}\to L_{-1}/L_1\to 0$ of vector bundles on X with $L_1\in\mathrm{Bun}_k$ and $L_{-1}/L_1\in\mathrm{Bun}_{2n-2k}$, and a symplectic form $\wedge^2(L_{-1}/L_1)\to\Omega$ (thus, $L_{-1}/L_1\in\mathrm{Bun}_{G_{n-k}}$).

Let $\eta_k:\mathrm{Bun}_{P_k}\to\mathrm{Bun}_{Q_k}$ denote the natural projection. Let ${}^0\mathrm{Bun}_{Q_k}\subset\mathrm{Bun}_{Q_k}$ be the open substack given by $\mathrm{H}^0(X,\mathrm{Sym}^2L_1)=0$.

Theorem 2. *For the diagram*

$$\widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{Q_k} \stackrel{\mathrm{id}\times\eta_k}{\longleftarrow} \widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{P_k} \stackrel{\tilde\nu_k}{\to} \widetilde{\mathrm{Bun}}_G$$

we have an isomorphism

$$(\mathrm{id}\times\eta_k)_!\tilde\nu_k^*\,\mathrm{Aut} \,\widetilde{\to}\, \mathrm{Aut}\boxtimes\bar{\mathbb{Q}}_\ell[b](\frac{b}{2})$$

over $\widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} {}^0\mathrm{Bun}_{Q_k}$. *(Once $\sqrt{-1}\in k$ is fixed, the isomorphism is well-defined up to a sign on generic and special parts). Here $b(L_1) = d_G - d_{G_{n-k}} - \chi(L_1) + 2\chi(\Omega^{-1}\otimes \mathrm{Sym}^2 L_1)$ is a function of a connected component of ${}^0\mathrm{Bun}_{Q_k}$. If $\chi(L_1)$ is even then, over the corresponding connected component, the above isomorphism preserves generic and special parts, othewise it interchanges them.*

3.4 In Sect. 8.1 we consider the affine Grassmanian Gr_G for G, it is equipped with a natural line bundle $\mathcal{L}$ that generates the Picard group of Gr_G. Let $\widetilde{\mathrm{Gr}}_G\to\mathrm{Gr}_G$ denote the μ_2-gerbe of square roots of $\mathcal{L}$. This is a local version of the gerbe (10). We introduce the category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\natural$ of *genuine spherical sheaves* on $\widetilde{\mathrm{Gr}}_G$ (cf. Definition 4 and 6).

As for usual spherical sheaves on the affine Grassmanian, we equip $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\natural$ with a structure of a rigid tensor category. Main result of Sect. 8 is the following version of the Satake equivalence.

Theorem 3. *The category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\natural$ is canonically equivalent, as a tensor category, to the category $\mathrm{Rep}(\mathbb{S}\mathrm{p}_{2n})$ of finite-dimensional $\bar{\mathbb{Q}}_\ell$-representations of $\mathbb{S}\mathrm{p}_{2n}$.*

In Sect. 9 we define for $K\in\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\natural$ Hecke operators $\mathrm{H}(K,\cdot):\mathrm{D}(\widetilde{\mathrm{Bun}}_G)\to\mathrm{D}(X\times\widetilde{\mathrm{Bun}}_G)$ compatible with the tensor structure on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\natural$. Finally, we prove Theorem 4 saying that Aut is a Hecke eigen-sheaf with eigenvalue

$$\mathrm{St} = \mathrm{R}\Gamma(\mathbb{P}^{2n-1},\bar{\mathbb{Q}}_\ell)\otimes\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes 2n-1}$$

viewed as a constant complex on X.

4. Finite-dimensional model

4.1 Let V be a k-vector space of dimension d. Write $\mathrm{ST}^2(V^*)$ for the space of symmetric tensors in $V^*\otimes V^*$, this is the space of symmetric bilinear forms on V. Think of $b\in\mathrm{ST}^2(V^*)$ as a map $b:V\to V^*$ such that $b^*=b$. Let $p:V\times\mathrm{ST}^2(V^*)\to\mathrm{ST}^2(V^*)$ denote the projection. Let $\beta: V\times\mathrm{ST}^2(V^*)\to\mathbb{A}^1$ be the map that sends (v,b) to $\langle v, bv\rangle$. Set

$$S_\psi = p_!\beta^*\mathcal{L}_\psi\otimes\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d+\frac{1}{2}d(d+1)}$$

Let $\pi: V\to\mathrm{Sym}^2 V$ be the map $v\mapsto v\otimes v$. Then

$$S_\psi = \mathrm{Four}_\psi(\pi_!\bar{\mathbb{Q}}_\ell[d](\frac{d}{2})) \tag{12}$$

The map π is finite, and $\pi_!\bar{\mathbb{Q}}_\ell = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is the constant sheaf on the image $\operatorname{Im}\pi$ of π, and $\mathcal{L}_1$ is a nontrivial local system of rank one on $\operatorname{Im}\pi - \{0\}$ extended by zero to $\operatorname{Im}\pi$. So, S_ψ is a direct sum of two irreducible perverse sheaves.

Lemma 2. *S_ψ is $\mathrm{GL}(V)$-equivariant.*

Proof Clearly, $\pi_!\bar{\mathbb{Q}}_\ell$ is $\mathrm{GL}(V)$-equivariant. The Fourier transform preserves $\mathrm{GL}(V)$-equivariance of a perverse sheaf. $\square$

Stratify $\mathrm{ST}^2(V^*)$ by $Q_i(V)$, where $Q_i(V)$ is the locus of $b : V \to V^*$ such that $\dim\operatorname{Ker} b = i$. For $b \in \mathrm{ST}^2(V^*)$ denote by $\beta_b : V \to \mathbb{A}^1$ the map $b \mapsto \langle v, bv\rangle$. We have a usual ambiguity in identifying $\mathrm{ST}^2(V^*)$ with $\mathrm{Sym}^2(V^*)$: b goes to β_b or $\frac{1}{2}\beta_b$. Since S_ψ is $\mathrm{GL}(V)$-equivariant, we can view it as a perverse sheaf on $\mathrm{Sym}^2(V^*)$ unambiguously.

Lemma 3. *For $b \in Q_0(V)$ the complex $\mathrm{R}\Gamma_c(V, \beta_b^*\mathcal{L}_\psi)$ is a 1-dimensional vector space placed in degree d.*

Proof In some basis β_b is given by $(x_1, \ldots, x_d) \mapsto x_1^2 + \ldots + x_d^2$. Thus we may assume $d = 1$. Consider the map $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $\pi(x) = x^2$. As above $\pi_!\bar{\mathbb{Q}}_\ell \widetilde{\to} \mathcal{L}_0 \oplus \mathcal{L}_1$ with $\mathcal{L}_0 = \bar{\mathbb{Q}}_\ell$. We get $\mathrm{R}\Gamma_c(\mathbb{A}^1, \pi^*\mathcal{L}_\psi) \widetilde{\to} \mathrm{R}\Gamma_c(\mathbb{G}_m, \mathcal{L} \otimes \mathcal{L}_\psi)$. The latter is a vector space of dimension one placed in degree one (a gamma-function on $\mathbb{G}_m$). $\square$

Let $\mathrm{Cov}(Q_0(V)) \to Q_0(V)$ denote the two-sheeted covering of $Q_0(V)$ whose fibre over $b : V \widetilde{\to} V^*$ is the set of trivializations $\det V \widetilde{\to} k$ whose square is the one induced by b.

The group $\mathrm{GL}(V)$ acts transitively on $Q_0(V)$, so given $b \in Q_0(V)$ one gets an identification $Q_0(V) \widetilde{\to} \mathrm{GL}(V)/\mathbb{O}(V, b)$. Our covering becomes the map $\mathrm{GL}(V)/\mathrm{S}\mathbb{O}(V, b) \to \mathrm{GL}(V)/\mathbb{O}(V)$.

More generally, $\mathrm{GL}(V)$ acts transitively on $Q_i(V)$. For $b \in Q_i(V)$ with $\operatorname{Ker} b = V_0$, we can consider b as an element of $\mathrm{Sym}^2(V/V_0)^*$. We get a parabolic $P_0 \subset \mathrm{GL}(V)$ of automorphisms of V that preserve V_0. Let $\mathrm{St}_{\mathrm{V}_0}$ be the preimage of $\mathbb{O}(V/V_0, b)$ under $P_0 \to \mathrm{GL}(V/V_0)$. Then $\mathrm{St}_{\mathrm{V}_0}$ is the stabilizer of $b \in Q_i(V)$ in $\mathrm{GL}(V)$. Since $\mathrm{S}\mathbb{O}(V, b)$ is connected, for $i < d$ there is exactly one (up to isomorphism) nonconstant $\mathrm{GL}(V)$-equivariant local system of rank one on $Q_i(V)$. It corresponds to the S_2-covering $\mathrm{Cov}(Q_i(V)) \to Q_i(V)$ whose fibre over b is the set of trivializations $\det(V/V_0) \widetilde{\to} k$ compatible with b.

Proposition 1. *1) The $*$-restriction of S_ψ to $Q_i(V)$ is a $\mathrm{GL}(V)$-equivariant local system of rank one placed in degree $i - \frac{1}{2}d(d+1)$. For $i < d$ this local system is nonconstant and comes from the covering $\mathrm{Cov}(Q_i(V)) \to Q_i(V)$.*
2) $S_\psi = S_{\psi,g} \oplus S_{\psi,s}$ is a direct sum of two irreducible perverse sheaves. Here $S_{\psi,g}$ is the Goresky-MacPherson extension of $S_\psi \mid_{Q_0(V)}$, and $S_{\psi,s}$ is the Goresky-MacPherson extension of $S_\psi \mid_{Q_1(V)}$ under $Q_1(V) \hookrightarrow Q_{\geq 1}(V)$.
3) We have $\mathbb{D}S_{\psi,g} \widetilde{\to} S_{\psi^{-1},g}$ and $\mathbb{D}S_{\psi,s} \widetilde{\to} S_{\psi^{-1},s}$ canonically.
4) If $V = V_1 \oplus V_2$ is a direct sum of two vector spaces of dimensions d_1 and d_2 then the $*$-restriction of $S_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes -\frac{1}{2}d(d+1)}$ to the subspace $\mathrm{Sym}^2(V_1^*) \oplus \mathrm{Sym}^2(V_2^*)$ is canonically

$$(S_\psi \boxtimes S_\psi) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes -\frac{1}{2}d_1(d_1+1) - \frac{1}{2}d_2(d_2+1)}$$

Proof 2) A point of $Q_i(V)$ is given by a subspace $V_0 \subset V$ of dimension i together with nondegenerate form $b : V/V_0 \to (V/V_0)^*$ such that $b^* = b$. It follows that

$$\dim Q_i(V) = \frac{1}{2}(d-i)(d+1-i) + (d-i)i = \frac{1}{2}(d-i)(d+1+i)$$

From Lemma 3 applied to V/V_0 we deduce that $S_\psi \mid_{Q_i(V)}$ is a local system of rank one placed in degree $i - \frac{1}{2}d(d+1)$. From (12) we see that $\mathbb{D}S_\psi \widetilde{\to} S_{\psi^{-1}}$. For $0 \le i \le d$ we have

$$\dim Q_i(V) = \frac{1}{2}(d-i)(d+1+i) \le \frac{1}{2}d(d+1) - i,$$

the equality holds only for $i = 0$ and $i = 1$. So, S_ψ is the Goresky-MacPherson extension from the open subscheme $Q_{\le 1}(V)$.

Let $S_{\psi,g}$ be the intermediate extension of $S_\psi \mid_{Q_0(V)}$ to $\operatorname{Sym}^2 V^*$. The $*$-restriction $S_{\psi,g} \mid_{Q_1(V)}$ vanishes. Indeed, it should be placed in strictly negative perverse degrees, but $S_\psi \mid_{Q_1(V)}$ is a perverse sheaf. Part 2) follows.

3) follows from (12)

4) The composition $V_1 \oplus V_2 \widetilde{\to} V \xrightarrow{\pi} \operatorname{Sym}^2 V \xrightarrow{a} \operatorname{Sym}^2 V_1 \times \operatorname{Sym}^2 V_2$ equals $\pi \times \pi$. So, $a_! \pi_! \bar{\mathbb{Q}}_\ell \widetilde{\to} (\pi_! \bar{\mathbb{Q}}_\ell \boxtimes \pi_! \bar{\mathbb{Q}}_\ell)$. Fourier transform interchanges $a_!$ and the $*$-restriction under the transpose map $a^* : \operatorname{Sym}^2 V_1^* \times \operatorname{Sym}^2 V_2^* \to \operatorname{Sym}^2 V^*$.

1) Since $S_\psi \mid_{Q_i(V)}$ is $\mathrm{GL}(V)$-equivariant, it remains to show it is nonconstant for $i < d$.

Step 1. Start with $d = 1$ case, so $Q_0(V) \widetilde{\to} \mathbb{G}_m$. To show that S_ψ is nonconstant on $Q_0(V)$ in this case, it suffices to prove that $\mathrm{R}\Gamma_c(\mathbb{G}_m, S_\psi) = 0$.

We will show that $\mathrm{R}\Gamma_c(\mathbb{A}^1 \times \mathbb{G}_m, \beta^* \mathcal{L}_\psi) = 0$, where the map $\beta : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1$ sends (v, b) to bv^2. Let $\tilde{\beta} : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1$ be the map that sends (v, b) to bv. For the projection $\mathrm{pr}_1 : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1$ we have

$$\mathrm{pr}_{1!} \tilde{\beta}^* \mathcal{L}_\psi \widetilde{\to} j_* \bar{\mathbb{Q}}_\ell[-1],$$

where $j : \mathbb{G}_m \to \mathbb{A}^1$ is the open immersion ([13], Lemma 2.3). Let $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ send v to v^2. From the diagram

$$\begin{array}{ccccc} \mathbb{A}^1 \times \mathbb{G}_m & \stackrel{\pi \times \mathrm{id}}{\to} & \mathbb{A}^1 \times \mathbb{G}_m & \stackrel{\tilde{\beta}}{\to} & \mathbb{A}^1 \\ \downarrow \mathrm{pr}_1 & & \downarrow \mathrm{pr}_1 & & \\ \mathbb{A}^1 & \stackrel{\pi}{\to} & \mathbb{A}^1 & & \end{array}$$

we learn that

$$\mathrm{pr}_{1!} \beta^* \mathcal{L}_\psi \widetilde{\to} \pi^* \mathrm{pr}_{1!} \tilde{\beta}^* \mathcal{L}_\psi$$

It suffices to show that $\mathrm{R}\Gamma_c(\mathbb{A}^1, \pi^* j_* \bar{\mathbb{Q}}_\ell) = 0$. Recall that $\pi_! \bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell \oplus \mathcal{L}_1$, where $\mathcal{L}_1$ is the local system on $\mathbb{G}_m$ extended by zero to $\mathbb{A}^1$, which corresponds to the Galois covering $\pi : \mathbb{G}_m \to \mathbb{G}_m$. We get

$$\mathrm{R}\Gamma_c(\mathbb{A}^1, \pi^* j_* \bar{\mathbb{Q}}_\ell) \widetilde{\to} \mathrm{R}\Gamma_c(\mathbb{A}^1, \pi_! \bar{\mathbb{Q}}_\ell \otimes j_* \bar{\mathbb{Q}}_\ell) = 0,$$

because $\mathrm{R}\Gamma_c(\mathbb{G}_m, \mathcal{L}_1) = 0$ and $\mathrm{R}\Gamma_c(\mathbb{A}^1, j_*\bar{\mathbb{Q}}_\ell) = 0$.

Step 2. For any d and $i < d$ choose a decomposition of V as a direct sum $V = W \oplus V_1 \oplus \ldots \oplus V_{d-i}$, where $\dim V_j = 1$ and $\dim W = i$. Then $Q_0(V_1) \times \ldots \times Q_0(V_{d-i}) \subset Q_i(V)$. The restriction of S_ψ to $Q_0(V_1) \times \ldots \times Q_0(V_{d-i})$ is nonconstant by Step 1 combined with 4). $\square$

Proposition 2. *A choice of a square root $i = \sqrt{-1} \in k$ yields for any j an isomorphism*

$$S_\psi \otimes S_\psi \mid_{Q_j(V)} \widetilde{\to} \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes -2j+d(d+1)}$$

Proof Let $\beta_2 : V \times V \times \mathrm{Sym}^2 V^* \to \mathbb{A}^1$ be the map sending (v, u, b) to $\langle v, bv\rangle + \langle u, bu\rangle$. Let $p_3 : V \times V \times \mathrm{Sym}^2 V^* \to \mathrm{Sym}^2 V^*$ be the projection. One checks that

$$S_\psi \otimes S_\psi \widetilde{\to} p_{3!}\beta_2^*\mathcal{L}_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes 2d+d(d+1)}$$

The change of variables

$$\left\{ \begin{array}{l} x = v + iu \\ y = v - iu \end{array} \right.$$

makes β_2 to be the map sending (x, y, b) to $\langle x, by\rangle$. Sommate first over x with y fixed, the assertion follows. $\square$

Proposition 3. *The $*$-restriction $\mathrm{Four}_\psi(\mathcal{L}_i) \mid_{Q_j(V)}$ vanishes if and only if $j \neq i + d \mod 2$. In other words, if $i = d \mod 2$ then $\mathrm{Four}_\psi(\mathcal{L}_i)$ has nontrivial fibres at $\cup_{j \text{ even}} Q_j(V)$. If $i \neq d \mod 2$ then $\mathrm{Four}_\psi(\mathcal{L}_i)$ has nontrivial fibres at $\cup_{j \text{ odd}} Q_j(V)$.*

In particular, $\mathrm{Four}_\psi(\mathcal{L}_i)[d](\frac{d}{2}) = S_{\psi,g}$ for $i = d \mod 2$ and $\mathrm{Four}_\psi(\mathcal{L}_i)[d](\frac{d}{2}) = S_{\psi,s}$ for $i \neq d \mod 2$.

Proof For $d = 1$ it is clear. Assume it is true for $d - 1$.

The complex $\mathrm{Four}_\psi(\mathcal{L}_j)$ is $\mathrm{GL}(V)$-equivariant, and $\mathrm{GL}(V)$ acts transitively on $Q_i(V)$. So, for each i exactly one of two sheaves $\mathrm{Four}_\psi(\mathcal{L}_0) \mid_{Q_i(V)}$ or $\mathrm{Four}_\psi(\mathcal{L}_1) \mid_{Q_i(V)}$ vanishes, and the other is a rank one (shifted) local system.

Write $V = V_1 \oplus V_2$, where $\dim V_1 = d - 1$ and $\dim V_2 = 1$. Consider the natural map $s : \mathrm{Sym}^2 V \to \mathrm{Sym}^2 V_1 \times \mathrm{Sym}^2 V_2$. We have

$$s_!(\mathcal{L}_0) \widetilde{\to} (\mathcal{L}_0 \boxtimes \mathcal{L}_0) \oplus (\mathcal{L}_1 \boxtimes \mathcal{L}_1)$$

and

$$s_!(\mathcal{L}_1) \widetilde{\to} (\mathcal{L}_0 \boxtimes \mathcal{L}_1) \oplus (\mathcal{L}_1 \boxtimes \mathcal{L}_0),$$

where on the right hand side $\mathcal{L}_i$ are those for V_1 and V_2.

Clearly, $Q_{i-1}(V_1) \times Q_1(V_2) \hookrightarrow Q_i(V)$ and $Q_i(V_1) \times Q_0(V_2) \hookrightarrow Q_i(V)$. Consider

$$\mathrm{Four}_\psi(\mathcal{L}_0) \mid_{Q_i(V_1) \times Q_0(V_2)} \widetilde{\to} h^*(\mathrm{Four}_\psi(\mathcal{L}_0) \boxtimes \mathrm{Four}_\psi(\mathcal{L}_0)) \oplus h^*(\mathrm{Four}_\psi(\mathcal{L}_1) \boxtimes \mathrm{Four}_\psi(\mathcal{L}_1)), \quad (13)$$

where $h : Q_i(V_1) \times Q_0(V_2) \hookrightarrow \mathrm{Sym}^2 V_1^* \times \mathrm{Sym}^2 V_2^*$. This isomorphism holds up to a shift and a twist.

If $i = d \mod 2$ then $h^*(\mathrm{Four}_\psi(\mathcal{L}_1) \boxtimes \mathrm{Four}_\psi(\mathcal{L}_1))$ is non zero by induction hypothesis, so the LHS of (13) does not vanish, hence $\mathrm{Four}_\psi(\mathcal{L}_0) \mid_{Q_i(V)}$ does not vanish either.

If $i \neq d \mod 2$ then the RHS of (13) vanishes by induction hypothesis, so the LHS also vanishes. Thus, $\mathrm{Four}_\psi(\mathcal{L}_0) \mid_{Q_i(V)}$ vanishes. $\square$

4.2 Assume $d \geq 1$. Let $Y(V)$ be the moduli scheme of pairs: a one dimensional subspace $V_0 \subset V$ and $b \in \mathrm{Sym}^2(V/V_0)^*$. The projection $Y(V) \to \mathrm{Gr}(1, V)$ is a vector bundle, where $\mathrm{Gr}(1, V)$ denotes the Grassmanian of one-dimensional subspaces in V. Let $\alpha : Y(V) \to \mathrm{Sym}^2 V^*$ be the map sending the above point to the composition

$$V \to V/V_0 \xrightarrow{b} (V/V_0)^* \hookrightarrow V^*$$

Clearly, α factors through $Q_{\geq 1}(V) \hookrightarrow \mathrm{Sym}^2 V^*$. Note that $Y(V)$ is smooth.

Proposition 4. *The map $\alpha : Y(V) \to Q_{\geq 1}(V)$ is proper surjective and semi-small.*

Proof Stratify $Q_{\geq 1}(V)$ by $Q_i(V)$ for $i \geq 1$. The fibre of α over a point $b \in Q_i(V)$ is the projective space of 1-subspaces in V', where V' is the kernel of b. So, $\dim \alpha^{-1}(b) = i - 1$ and $\dim Q_i(V) = \frac{1}{2}(d - i)(d + 1 + i)$. We get

$$2 \dim \alpha^{-1}(b) \leq \mathrm{codim}_{Q_{\geq 1}(V)} Q_i(V),$$

the equality holds only for $i = 1, 2$. $\square$

4.3 Relative version Let now S be a smooth scheme, $V \to S$ be a vector bundle of rank d. Define $S_\psi \in \mathrm{D}(\mathrm{Sym}^2 V^*)$ by (12), so S_ψ is a shifted perverse sheaf.

As above, $\mathrm{Sym}^2 V^*$ is stratified by locally closed subschemes $Q_i(V)$, they are equipped with morphisms $Q_i(V) \to \mathrm{Gr}(i, V)$ over S.

We also have the S_2-coverings $\mathrm{Cov}(Q_i(V)) \to Q_i(V)$. For an S-scheme S', the S'-points of $\mathrm{Cov}(Q_i(V))$ are collections: a rank i subbundle $V_0 \subset V \mid_{S'}$, an isomorphism $b : V/V_0 \to (V/V_0)^*$ of $\mathcal{O}_{S'}$-modules with $b^* = b$, and a compatible trivialization $\det(V/V_0) \widetilde{\to} \mathcal{O}_{S'}$.

Propositions 1, 2 and 3 hold in relative situation (one only changes a shift and a twist in 3) of Proposition 1).

4.4 Finite-dimensional theta-sheaf This subsection is not used in the proofs and may be skipped.

Let M be a symplectic k-vector space of dimension $2d$. Write $\mathcal{L}(M)$ for the scheme of lagrangian subspaces of M. Set $Y = \mathcal{L}(M) \times \mathcal{L}(M)$. Consider the line bundle $\mathcal{A}$ on Y with fibre $(\det L_1) \otimes (\det L_2)$ over $(L_1, L_2) \in Y$. We view it as $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero. Let $\tilde{Y}$ denote the stack of square roots of $\mathcal{A}$. The μ_2-gerbe $\tilde{Y} \to Y$ is nontrivial. The group $\mathbb{S}\mathrm{p}(M)$ acts naturally on Y, and $\mathcal{A}$ is $\mathbb{S}\mathrm{p}(M)$-equivariant, so $\mathbb{S}\mathrm{p}(M)$ acts also on $\tilde{Y}$.

We are going to construct a $\mathbb{S}\mathrm{p}(M)$-equivariant perverse sheaf S_M on $\tilde{Y}$ such that $-1 \in \mu_2$ acts on S_M as -1.

The $\mathbb{S}\mathrm{p}(M)$-orbits on Y are indexed by $i = 0, \dots, d$. The orbit Y_i is given by $\dim(L_1 \cap L_2) = i$.

Lemma 4. *The restriction of $\mathcal{A}$ to each Y_i admits a canonical $\mathbb{S}\mathrm{p}(M)$-equivariant square root.*

Proof For $L_1, L_2 \in \mathcal{L}(M)$ let $(L_1 \cap L_2)^\perp \subset M$ denotes the orthogonal complement to $L_1 \cap L_2$. The symplectic form on $(L_1 \cap L_2)^\perp/(L_1 \cap L_2)$ induces an isomorphism $L_2/(L_1 \cap L_2) \widetilde{\to} (L_1/L_1 \cap L_2)^*$. This yields a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $(\det L_1) \otimes (\det L_2) \widetilde{\to} \det(L_1 \cap L_2)^{\otimes 2}$. By 3.1.2, we are done. $\square$

Let W denote the nontrivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\operatorname{Spec} k \to B(\mu_2)$. Let $\tilde{Y}_i$ denote the restriction of the gerbe $\tilde{Y} \to Y$ to Y_i, so $\tilde{Y}_i \widetilde{\to} Y_i \times B(\mu_2)$ canonically.

Definition 2. Let $S_{M,g}$ (resp., $S_{M,s}$) denote the Goresky-MacPherson extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W)[\dim Y](\frac{\dim Y}{2})$$

from $\tilde{Y}_0$ to $\tilde{Y}$ (resp., of $(\bar{\mathbb{Q}}_\ell \boxtimes W)[\dim Y - 1](\frac{\dim Y - 1}{2})$ from $\tilde{Y}_1$ to $\tilde{Y}$). Set $S_M = S_{M,g} \oplus S_{M,s}$.

Denote by $\mathcal{Y}$ the stack quotient $Y/\mathbb{S}\mathrm{p}(M)$. Write $\tilde{\mathcal{Y}} \to \mathcal{Y}$ for the corresponding gerbe of square roots of $\mathcal{A}$. We may view S_M as a perverse sheaf on $\tilde{\mathcal{Y}}$.

Fix a lagrangian subspace $V \subset M$, let $P_V \subset \mathbb{S}\mathrm{p}(M)$ be the Seigel parabolic subgroup preserving V. We have canonical isomorphisms of stacks

$$\mathcal{Y} \widetilde{\to} \mathcal{L}(M)/P_V \widetilde{\to} P_V \backslash \mathbb{S}\mathrm{p}(M)/P_V$$

One may view $\mathcal{A}$ as a line bundle on $\mathcal{L}(M)/P_V$ with fibre $(\det V) \otimes (\det L)$.

Fix a splitting $V^* \to M$ of $0 \to V \to M \to V^* \to 0$. Denote by $P_V^- \subset \mathbb{S}\mathrm{p}(M)$ the Seigel parabolic subgroup preserving $V^* \subset M$. Let $Z \subset \mathcal{L}(M)$ be the open P_V^--orbit, that is

$$Z = \{L \in \mathcal{L}(M) \mid L \cap V^* = 0\}$$

The map $\operatorname{Sym}^2 V^* \to Z$ sending $b : V \to V^*$ to $L = \{v + bv \in M \mid v \in V\}$ is an isomorphism commuting with the action of $\mathrm{GL}(V)$. Denote by $\mathcal{Z}$ the stack quotient $Z/\mathrm{GL}(V)$. View S_ψ introduced in Sect. 4.1 as a perverse sheaf on $\mathcal{Z}$.

Denote by ν the composition (of an open immersion followed by a smooth map)

$$\mathcal{Z} \hookrightarrow \mathcal{L}(M)/\mathrm{GL}(V) \to \mathcal{L}(M)/P_V = \mathcal{Y}$$

The map $\nu : \mathcal{Z} \to \mathcal{Y}$ is smooth, surjective and representable. It factors naturally as $\mathcal{Z} \xrightarrow{\tilde{\nu}} \tilde{\mathcal{Y}} \to \mathcal{Y}$.

Proposition 5. *There are isomorphisms of perverse sheaves on $\mathcal{Z}$*

$$\tilde{\nu}^* S_{M,g} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{Z} - \dim \mathcal{Y}} \widetilde{\to} S_{\psi,g}$$

and

$$\tilde{\nu}^* S_{M,s} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{Z} - \dim \mathcal{Y}} \widetilde{\to} S_{\psi,s}$$

(Once $i = \sqrt{-1} \in k$ is fixed, such isomorphisms are well defined up to multiplication by ± 1).

Proof The stack $\mathcal{Z}$ is stratified by $\mathcal{Z}_i = Q_i(V)/\operatorname{GL}(V)$, the quotient being taken in stack sense. Let $\mathcal{Y}_i$ denote the stack quotient $Y_i/\mathbb{S}\mathrm{p}(M)$. Note that $\mathcal{Z}_i$ identifies with $\mathcal{Z}\times_{\mathcal{Y}}\mathcal{Y}_i$ for $i=0,\dots,d$.

Let $\tilde{\mathcal{Y}}_i$ denote the restriction of the gerbe $\tilde{\mathcal{Y}}$ to $\mathcal{Y}_i$, so $\tilde{\mathcal{Y}}_i \,\widetilde{\to}\, \mathcal{Y}_i \times B(\mu_2)$ canonically. Remind the covering $\operatorname{Cov}(Q_i(V)) \to Q_i(V)$ from Sect. 4.1. It is $\operatorname{GL}(V)$-equivariant, so the stack quotient $\operatorname{Cov}(\mathcal{Z}_i) = \operatorname{Cov}(Q_i(V))/\operatorname{GL}(V)$ is a two-sheeted covering of $\mathcal{Z}_i$. For each i we have a cartesian square

$$\begin{array}{ccc} \operatorname{Cov}(\mathcal{Z}_i) & \to & \mathcal{Y}_i \\ \downarrow & & \downarrow \\ \mathcal{Z}_i & \stackrel{\tilde{\nu}}{\to} & \tilde{\mathcal{Y}}_i \end{array}$$

Our assertion follows now from Proposition 1. $\square$

Remark 2. Write ${}_MY$ (resp., ${}_M\tilde{\mathcal{Y}}$) to express the dependence on M. If M, M' are two symplectic spaces over k of dimensions d, d', consider the map $\tau_{M,M'} : {}_MY \times {}_{M'}Y \to {}_{M\oplus M'}Y$ sending $(L_1, L_2), (L'_1, L'_2)$ to $(L_1 \oplus L'_1, L_2 \oplus L'_2)$. It yields a map

$$\tilde{\tau}_{M,M'} : {}_M\tilde{\mathcal{Y}} \times {}_{M'}\tilde{\mathcal{Y}} \to {}_{M\oplus M'}\tilde{\mathcal{Y}}$$

From 4) of Proposition 1 it follows that $\tilde{\tau}^*_{M,M'} S_{M\oplus M'} \,\widetilde{\to}\, S_M \boxtimes S_{M'}[2dd'](dd')$ canonically.

5. Fourier coefficients of Aut for Siegel parabolic

5.1 Write $\operatorname{Bun}_P = \operatorname{Bun}_{P_n}$. So, Bun_P classifies $L \in \operatorname{Bun}_n$ together with an exact sequence $0 \to \operatorname{Sym}^2 L \to ? \to \Omega \to 0$ on X. It induces an exact sequence

$$0 \to L \to M \to L^* \otimes \Omega \to 0, \tag{14}$$

The map $\nu_n : \operatorname{Bun}_P \to \operatorname{Bun}_G$ is also denoted ν.

Lemma 5. *The map* $\nu : \operatorname{Bun}_P \to \operatorname{Bun}_G$ *factors as the composition* $\operatorname{Bun}_P \stackrel{\tilde{\nu}}{\to} \widetilde{\operatorname{Bun}}_G \stackrel{\mathfrak{r}}{\to} \operatorname{Bun}_G$.

Proof The sequence (14) yields a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det \mathrm{R}\Gamma(X, M) \,\widetilde{\to}\, \det \mathrm{R}\Gamma(X, L) \otimes \det \mathrm{R}\Gamma(X, L^* \otimes \Omega) \,\widetilde{\to}\, \det \mathrm{R}\Gamma(X, L^* \otimes \Omega)^2 \tag{15}$$

Define $\tilde{\nu}$ by letting $\mathcal{B} = \det \mathrm{R}\Gamma(X, L^* \otimes \Omega)$ together with $\mathcal{B}^2 \,\widetilde{\to}\, \mathcal{A}$ given by (15). By 3.1.2, $\tilde{\nu}$ is well-defined. $\square$

Let ${}^0\operatorname{Bun}_P \subset \operatorname{Bun}_P$ be the open substack given by $\mathrm{H}^0(X, \operatorname{Sym}^2 L) = 0$. One checks that both $\nu : {}^0\operatorname{Bun}_P \to \operatorname{Bun}_G$ and $\tilde{\nu} : {}^0\operatorname{Bun}_P \to \widetilde{\operatorname{Bun}}_G$ are smooth.

Lemma 6. *The map* $\nu : {}^0\operatorname{Bun}_P \to \operatorname{Bun}_G$ *is surjective, so* $\tilde{\nu} : {}^0\operatorname{Bun}_P \to \widetilde{\operatorname{Bun}}_G$ *is also surjective.*

Proof Let M be a k-point of Bun_G. It admits a line subbundle L_1 with $\deg L_1 < 0$. Let $L_{-1} \subset M$ be the orthogonal complement to L_1, so $L_{-1}/L_1 \in \mathrm{Bun}_{G_{n-1}}$ naturally. Continuing this procedure for L_{-1}/L_1 and so on, we get a flag of isotropic subbundles $L_1 \subset \ldots \subset L_n \subset M$. Then $(L_n \subset M)$ is a k-point of ${}^0\mathrm{Bun}_P$. $\square$

5.2 THE SHEAF $S_{P,\psi}$ ON Bun_P

Write Bun_n^d (resp., Bun_P^d) for the connected component of the corresponding stack given by $\deg L = d$.

Write ${}_c\mathrm{Bun}_n \subset \mathrm{Bun}_n$ for the open substack given by $\mathrm{H}^0(X, L) = 0$. Let $\mathcal{V} \to \mathrm{Bun}_n$ be the stack whose fibre over $L \in \mathrm{Bun}_n$ is $\mathrm{Hom}(L, \Omega)$. Let ${}_c\mathcal{V} \to {}_c\mathrm{Bun}_n$ be the preimage of ${}_c\mathrm{Bun}_n$, over ${}_c\mathrm{Bun}_n^d$ this is a vector bundle of rank $n(g-1)-d$.

Let $\mathcal{X} = \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_P$ and $p : \mathcal{X} \to \mathrm{Bun}_P$ be the projection. Let $q : \mathcal{X} \to \mathbb{A}^1$ be the map sending $s \in \mathrm{H}^0(X, L^* \otimes \Omega)$ to the pairing of $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$ with

$$s \otimes s \in \mathrm{H}^0(X, (\mathrm{Sym}^2 L^*) \otimes \Omega^2)$$

Definition 3. Set $S_{P,\psi} = p_! q^* \mathcal{L}_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{X}}$, where $\dim \mathcal{X}$ is the dimension of the corresponding connected component of $\mathcal{X}$.

Let $\mathcal{V}_2 \to \mathrm{Bun}_n$ be the stack whose fibre over $L \in \mathrm{Bun}_n$ is $\mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$. Let $\pi_2 : \mathcal{V} \to \mathcal{V}_2$ be the map sending $s \in \mathrm{Hom}(L, \Omega)$ to $s \otimes s$. Note that π_2 is finite, a S_2-covering over the image $\mathrm{Im}\, \pi_2$ with removed zero section. By definition,

$$S_{P,\psi} \widetilde{\to} \mathrm{Four}_\psi(\pi_{2!}\bar{\mathbb{Q}}_\ell) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{V}}, \tag{16}$$

where $\mathrm{Four}_\psi : \mathrm{D}(\mathcal{V}_2) \to \mathrm{D}(\mathrm{Bun}_P)$ is the Fourier transform functor. Note that S_2 acts on $S_{P,\psi}$.

Let ${}_c\mathrm{Bun}_P$ denote the preimage of ${}_c\mathrm{Bun}_n$ in Bun_P. We see that over each connected component of ${}_c\mathrm{Bun}_P$, $S_{P,\psi}$ is a direct sum of two irreducible perverse sheaves and $\mathbb{D}(S_{P,\psi}) \widetilde{\to} S_{P,\psi^{-1}}$.

Let $\mathrm{Sym}^2 {}_c\mathcal{V} \to {}_c\mathrm{Bun}_n$ denote the symmetric square of the vector bundle ${}_c\mathcal{V} \to {}_c\mathrm{Bun}_n$. Let $\pi : {}_c\mathcal{V} \to \mathrm{Sym}^2 {}_c\mathcal{V}$ be the map sending $s \in \mathrm{Hom}(L, \Omega)$ to $s \otimes s$. Then π_2 decomposes as

$${}_c\mathcal{V} \xrightarrow{\pi} \mathrm{Sym}^2 {}_c\mathcal{V} \xrightarrow{f^*} {}_c\mathcal{V}_2$$

Given $L \in \mathrm{Bun}_n$, the transpose to the linear map $\mathrm{Sym}^2 \mathrm{H}^0(X, L^* \otimes \Omega) \to \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$ is

$$\mathrm{H}^1(X, (\mathrm{Sym}^2 L) \otimes \Omega^{-1}) \to \mathrm{Sym}^2 \mathrm{H}^1(X, L)$$

It defines a morphism of stacks $f : {}_c\mathrm{Bun}_P \to \mathrm{Sym}^2 {}_c\mathcal{V}^*$ over ${}_c\mathrm{Bun}_n$.

We have the sheaf S_ψ on $\mathrm{Sym}^2 {}_c\mathcal{V}^*$ defined in Sect. 4.3. From (16) we conclude that

$$S_{P,\psi} \widetilde{\to} f^* S_\psi \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{X} - r - \frac{1}{2}r(r+1)} \tag{17}$$

canonically over ${}_c\mathrm{Bun}_P$, where r and $\dim \mathcal{X}$ are functions of the corresponding connected component with $r({}_c\mathrm{Bun}_P^d) = n(g-1)-d$.

Denote by $S_{P,\psi,g}$ (resp., by $S_{P,\psi,s}$) the direct summand of $S_{P,\psi}$ obtained by replacing S_ψ by $S_{\psi,g}$ (resp., by $S_{\psi,s}$) in (17). Both $S_{P,\psi,g}$ and $S_{P,\psi,s}$ are irreducible perverse sheaves over each connected component of ${}_c\operatorname{Bun}_P$.

Note that ${}^0\operatorname{Bun}_P \subset {}_c\operatorname{Bun}_P$.

Remark 3. Consider $\chi(L)$ as a function of a connected component of ${}_c\operatorname{Bun}_P$. By Proposition 3, over a given connected component of ${}_c\operatorname{Bun}_P$, the S_2-invariants of $S_{P,\psi}$ are $S_{P,\psi,g}$ for $\chi(L)$ even and $S_{P,\psi,s}$ for $\chi(L)$ odd.

5.3 Recall the stratification of $\operatorname{Sym}^2 {}_c\mathcal{V}^*$ by locally closed substacks $Q_i({}_c\mathcal{V})$ and the coverings $\operatorname{Cov}(Q_i({}_c\mathcal{V})) \to Q_i({}_c\mathcal{V})$ defined in Sect. 4.3.

Set ${}_i\operatorname{Bun}_P = \nu^{-1}({}_i\operatorname{Bun}_G)$ and ${}_{i,c}\operatorname{Bun}_P = {}_c\operatorname{Bun}_P \cap {}_i\operatorname{Bun}_P$. For a point of ${}_c\operatorname{Bun}_P$ the exact sequence (14) yields an exact sequence

$$0 \to \mathrm{H}^0(X, M) \to \mathrm{H}^0(X, L^* \otimes \Omega) \xrightarrow{b} \mathrm{H}^1(X, L) \to \mathrm{H}^1(X, M) \to 0 \tag{18}$$

Thus, we get a commutative diagram

$$\begin{array}{ccc} {}_{i,c}\operatorname{Bun}_P & \hookrightarrow & {}_c\operatorname{Bun}_P \\ \downarrow f & & \downarrow f \\ Q_i({}_c\mathcal{V}) & \hookrightarrow & \operatorname{Sym}^2 {}_c\mathcal{V}^* \end{array}$$

Let ${}_i\rho_P : \operatorname{Cov}({}_{i,c}\operatorname{Bun}_P) \to {}_{i,c}\operatorname{Bun}_P$ be the covering obtained from $\operatorname{Cov}(Q_i({}_c\mathcal{V})) \to Q_i({}_c\mathcal{V})$ by the base change $f : {}_{i,c}\operatorname{Bun}_P \to Q_i({}_c\mathcal{V})$.

Proposition 6. *For $i \geq 0$ there is a cartesian square*

$$\begin{array}{ccc} \operatorname{Cov}({}_{i,c}\operatorname{Bun}_P) & \to & \operatorname{Cov}({}_i\widetilde{\operatorname{Bun}}_G) \\ \downarrow {}_i\rho_P & & \downarrow {}_i\rho \\ {}_{i,c}\operatorname{Bun}_P & \xrightarrow{\tilde{\nu}} & {}_i\widetilde{\operatorname{Bun}}_G \end{array}$$

Proof Let S be a scheme. Assume given an S-point of ${}_{i,c}\operatorname{Bun}_P$. It yields locally free $\mathcal{O}_S$-modules $V_0 = \mathrm{H}^0(X, M)$ and $V = \mathrm{H}^0(X, L^* \otimes \Omega)$ included into an exact sequence of $\mathcal{O}_S$-modules (a relative version of (18))

$$0 \to V_0 \to V \xrightarrow{b} V^* \to V_0^* \to 0$$

with $b^* = b$. The $\mathcal{O}_{S\times X}$-module L together with the morphism of $\mathcal{O}_S$-modules $V \xrightarrow{b} V^*$ defines the corresponding S-point of $Q_i({}_c\mathcal{V})$.

We have an isomorphism of $\mathcal{O}_S$-modules $\mathcal{B} = \det \mathrm{R}\Gamma(X, L^* \otimes \Omega) \widetilde{\to} \det V$, because $\mathrm{H}^0(X, L) = 0$. We also have an isomorphism of $\mathcal{O}_S$-modules $t : \mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M) \widetilde{\to} (\det V_0)^2$ given by (15).

A lifting of the corresponding S-point of ${}_i\widetilde{\operatorname{Bun}}_G$ to $\operatorname{Cov}({}_i\widetilde{\operatorname{Bun}}_G)$ is an isomorphism of $\mathcal{O}_S$-modules $\mathcal{B} \widetilde{\to} \det V_0$ whose square is t. The corresponding category is the category of S-points of $\operatorname{Cov}({}_{i,c}\operatorname{Bun}_P)$. $\square$

Proposition 7. *There are isomorphisms of perverse sheaves on* ${}^0\mathrm{Bun}_P$

$$\tilde{\nu}^* \mathrm{Aut}_g \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathrm{Bun}_P - d_G} \widetilde{\to} S_{P,\psi,g}$$

and

$$\tilde{\nu}^* \mathrm{Aut}_s \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathrm{Bun}_P - d_G} \widetilde{\to} S_{P,\psi,s}$$

Here $\dim \mathrm{Bun}_P$ *denotes the dimension of the corresponding connected component of* Bun_P. *(Once* $\sqrt{-1} \in k$ *is fixed, the above isomorphisms are well-defined up to a sign).*

Proof Recall that $S_{P,\psi,g}$ and $S_{P,\psi,s}$ are irreducible perverse sheaves over each connected component of ${}_c\mathrm{Bun}_P$. By relative version of Proposition 1, $S_{P,\psi,g}$ over ${}_{0,c}\mathrm{Bun}_P$ (resp., $S_{P,\psi,s}$ over ${}_{1,c}\mathrm{Bun}_P$) is a nonconstant local system of rank one corresponding to the covering $\mathrm{Cov}({}_{0,c}\mathrm{Bun}_P) \to {}_{0,c}\mathrm{Bun}_P$ (resp., $\mathrm{Cov}({}_{1,c}\mathrm{Bun}_P) \to {}_{1,c}\mathrm{Bun}_P$). Moreover, for any i

$$(S_{P,\psi} \otimes S_{P,\psi}) \mid_{i,c\,\mathrm{Bun}_P} \widetilde{\to} \bar{\mathbb{Q}}_\ell[2](1)^{\otimes \dim \mathrm{Bun}_P - i}$$

by Proposition 2 (this requires a choice of $\sqrt{-1} \in k$).

By Proposition 6, for each i we get isomorphisms

$$\tilde{\nu}^*({}_i\mathrm{Aut}) \mid_{i,c\,\mathrm{Bun}_P} \widetilde{\to} \mathrm{Hom}_{S_2}(\mathrm{sign}, ({}_i\rho_P)_!\bar{\mathbb{Q}}_\ell))$$

In particular,

$$\tilde{\nu}^*({}_i\mathrm{Aut} \otimes_i \mathrm{Aut}) \mid_{i,c\,\mathrm{Bun}_P} \widetilde{\to} \bar{\mathbb{Q}}_\ell$$

Set ${}^0_i\mathrm{Bun}_P = {}^0\mathrm{Bun}_P \cap {}_i\mathrm{Bun}_P$. By construction, $S_{P,\psi,s}$ is perverse over ${}_{1,c}\mathrm{Bun}_P$, hence also over ${}^0_1\mathrm{Bun}_P$. Since ${}^0_1\mathrm{Bun}_P \to {}_1\mathrm{Bun}_G$ is smooth and surjective, Propositions 1 and 6 imply that ${}_1\mathrm{Aut} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_G - 1}$ is perverse on ${}_1\mathrm{Bun}_G$. So, Defenition 1 makes sense.

For each connected component ${}^0\mathrm{Bun}_P^d$ of ${}^0\mathrm{Bun}_P$ the stack ${}^0\mathrm{Bun}_P^d \cap {}_i\mathrm{Bun}_P$ is non empty for $i = 0, 1$. Since $\tilde{\nu} : {}^0\mathrm{Bun}_P \to \widetilde{\mathrm{Bun}}_G$ is smooth, our assertion follows. $\square$

Proof of Theorem 1 For each d the map $\tilde{\nu} : {}^0\mathrm{Bun}_P^d \to \widetilde{\mathrm{Bun}}_G$ is smooth with connected fibres, and $\tilde{\nu} : {}^0\mathrm{Bun}_P \to \widetilde{\mathrm{Bun}}_G$ is surjective. So, by Proposition 7 it suffices to construct isomorphisms

$$S_{P,\psi} \mid_{{}^0_i\mathrm{Bun}_P} \widetilde{\to} \tilde{\nu}^*({}_i\mathrm{Aut}) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathrm{Bun}_P - i}$$

over ${}^0_i\mathrm{Bun}_P$. We have them by Proposition 6 combined with relative version of Proposition 1. Proposition 3 implies the second part of the theorem. $\square$

Remark 4. From Theorem 1 it follows that $\tilde{\nu}^* \mathrm{Aut} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathrm{Bun}_P - d_G}$ equals $S_{P,\psi}$ in the Grothendieck group $K({}_c\mathrm{Bun}_P)$ over ${}_c\mathrm{Bun}_P$, which is bigger than ${}^0\mathrm{Bun}_P$. We expect that actually the isomorphisms of Proposition 7 hold over ${}_c\mathrm{Bun}_P$.

6. Constant terms of Aut for maximal parabolics

6.1 Recall the smooth map $\eta_k : \mathrm{Bun}_{P_k} \to \mathrm{Bun}_{Q_k}$ (cf. Sect. 3.3). Under each of the two projections $\mathrm{Bun}_{P_k} \times_{\mathrm{Bun}_{Q_k}} \mathrm{Bun}_{P_k} \to \mathrm{Bun}_{P_k}$ the stack $\mathrm{Bun}_{P_k} \times_{\mathrm{Bun}_{Q_k}} \mathrm{Bun}_{P_k}$ identifies with the one classifying $(L_1 \subset L_{-1} \subset M) \in \mathrm{Bun}_{P_k}$ together with an exact sequence $0 \to \mathrm{Sym}^2 L_1 \to ? \to \Omega \to 0$, the projection being the forgetful map.

Let $\nu_{k,n} : \mathrm{Bun}_{P_{k,n}} \to \mathrm{Bun}_P$ be the stack classifying $(0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0) \in \mathrm{Bun}_P$ together with a subbundle $L_1 \subset L$ with $L_1 \in \mathrm{Bun}_k$.

Lemma 7. *The map $\eta_k : \mathrm{Bun}_{P_k} \to \mathrm{Bun}_{Q_k}$ is surjective.*

Proof Consider a k-point of Bun_{Q_k} given by a flag $(L_1 \subset L_{-1})$ of vector bundles on X with $L_{-1}/L_1 \in \mathrm{Bun}_{G_{n-k}}$. Let show that the fibre of η_k over it is nonempty.

Pick a lagrangian subbundle $\mathcal{B} \subset L_{-1}/L_1$ such that $\mathrm{H}^1(X, \mathcal{B}^* \otimes L_1) = 0$, it always exists. Let $L \subset L_{-1}$ be the preimage of $\mathcal{B}$ under $L_{-1} \to L_{-1}/L_1$. The exact sequence $0 \to L_1 \to L \to \mathcal{B} \to 0$ splits, we fix a splitting $L \widetilde{\to} L_1 \oplus \mathcal{B}$. Then our k-point of Bun_{Q_k} becomes the data of two exact sequences

$$0 \to \mathrm{Sym}^2 \mathcal{B} \to ? \to \Omega \to 0$$

and

$$0 \to L_1 \to ? \to \mathcal{B}^* \otimes \Omega \to 0,$$

Pick any exact sequence $0 \to \mathrm{Sym}^2 L_1 \to ? \to \Omega \to 0$ and summate it with the above two. The result is an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$, the corresponding $P_{k,n}$-torsor induces a P_k-torsor lying in the fibre under consideration. $\square$

Set $\mathrm{Bun}_{Q_{k,n}} = \mathrm{Bun}_{P(G_{n-k})} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{Q_k}$, where $P(G_{n-k}) \subset G_{n-k}$ is the Siegel parabolic. So, $\mathrm{Bun}_{Q_{k,n}}$ classifies a point $0 \to L_1 \to L_{-1} \to L_{-1}/L_1 \to 0$ of Bun_{Q_k} together with a lagrangian subbundle $L/L_1 \subset L_{-1}/L_1$. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Bun}_P & \stackrel{\nu_{k,n}}{\leftarrow} & \mathrm{Bun}_{P_{k,n}} & \stackrel{\eta_{k,n}}{\to} & \mathrm{Bun}_{Q_{k,n}} \\
 & & & & \downarrow r_k \\
 & & & & \mathrm{Bun}_{P(G_{n-k})},
\end{array}$$

where r_k and $\eta_{k,n}$ denote the projections.

Write $S_{P(G_n),\psi}$ to express the dependence of $S_{P,\psi}$ on n. Note that $\mathrm{Bun}_{P(G_0)} = \mathrm{Spec}\, k$, $S_{P(G_0),\psi,g} = \bar{\mathbb{Q}}_\ell$ and $S_{P(G_0),\psi,s} = 0$.

Proposition 8. *We have a canonicall isomorphism commuting with S_2-action*

$$(\eta_{k,n})_! \nu_{k,n}^* S_{P,\psi} \widetilde{\to} r_k^* S_{P(G_{n-k}),\psi}[a](\frac{a}{2}),$$

where $a \in \mathbb{Z}$ is the function of a connected component of $\mathrm{Bun}_{Q_{k,n}}$ given by

$$a = \dim \mathrm{Bun}_n - \dim \mathrm{Bun}_{n-k} - \chi(L_1) + \chi(\Omega^{-1} \otimes \mathrm{Sym}^2 L_1) - \chi(\Omega^{-1} \otimes L_1 \otimes (L/L_1))$$

Proof Consider the map

$$\mathcal{X} \times_{\mathrm{Bun}_P} \mathrm{Bun}_{P_{k,n}} = \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{P_{k,n}} \stackrel{\mathrm{id}\times\eta_{k,n}}{\to} \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{Q_{k,n}}$$

Write $\mathbb{A}^1 \stackrel{q_n}{\leftarrow} \mathcal{X}_{G_n} \stackrel{p_n}{\to} \mathrm{Bun}_{P(G_n)}$ to express the dependence on n of the diagram $\mathbb{A}^1 \stackrel{q}{\leftarrow} \mathcal{X} \stackrel{p}{\to} \mathrm{Bun}_P$ introduced in Sect. 5.2.

Denote temporary by $i : \mathcal{X}_{G_{n-k}} \times_{\mathrm{Bun}_{P(G_{n-k})}} \mathrm{Bun}_{Q_{k,n}} \hookrightarrow \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{Q_{k,n}}$ the closed embedding given by the condition that $s \in \mathrm{Hom}(L, \Omega)$ lies in $\mathrm{Hom}(L/L_1, \Omega)$.

Set $a_0 = -\chi(\Omega^{-1} \otimes \mathrm{Sym}^2 L_1)$ viewed as a function of a connected component of $\mathrm{Bun}_{Q_{k,n}}$. Let us establish a canonical isomorphism

$$(\mathrm{id}\times\eta_{k,n})_!(q^*\mathcal{L}_\psi \boxtimes \bar{\mathbb{Q}}_\ell) \widetilde{\to} i_!(q_{n-k}^*\mathcal{L}_\psi \boxtimes \bar{\mathbb{Q}}_\ell)[-2a_0](-a_0) \tag{19}$$

Consider a k-point of $\mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{P_{k,n}}$ given by $(L_1 \subset L \subset L_{-1} \subset M)$ and $s \in \mathrm{Hom}(L, \Omega)$. The fibre, say Y, of $\mathrm{id}\times\eta_{k,n}$ over its image in $\mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{Q_{k,n}}$ identifies with the stack of exact sequences

$$0 \to \mathrm{Sym}^2 L_1 \to ? \to \Omega \to 0 \tag{20}$$

on X. The restriction of $q^*\mathcal{L}_\psi \boxtimes \bar{\mathbb{Q}}_\ell$ to Y is (up to a tensoring by a 1-dimensional vector space) the restriction of $\mathcal{L}_\psi$ under the map $Y \to \mathbb{A}^1$ pairing $\mathrm{Sym}^2 L_1 \hookrightarrow \mathrm{Sym}^2 L \stackrel{s\otimes s}{\to} \Omega^2$ with (20).

So, the fibre of the LHS of (19) vanishes unless $s \in \mathrm{Hom}(L/L_1, \Omega)$. The isomorphism (19) follows, here $a_0 = \dim Y$.

For the projection $\mathrm{pr} : \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{Q_{k,n}} \to \mathrm{Bun}_{Q_{k,n}}$ we get

$$\mathrm{pr}_!(\mathrm{id}\times\eta_{k,n})_!(q^*\mathcal{L}_\psi \boxtimes \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathcal{X}} \widetilde{\to} (\eta_{k,n})_!\nu_{k,n}^* S_{P,\psi}$$

Our assertion follows, because $a = \dim \mathcal{X}_{G_n} - \dim \mathcal{X}_{G_{n-k}} - 2a_0$. $\square$

Proof of Theorem 2 We have the diagram

$$\begin{array}{ccc}
\mathrm{Bun}_P & \stackrel{\tilde\nu}{\to} & \widetilde{\mathrm{Bun}}_G \\
\uparrow \nu_{k,n} & & \uparrow \tilde\nu_k \\
\mathrm{Bun}_{P_{k,n}} & \to & \widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{P_k} \\
\downarrow \eta_{k,n} & & \downarrow \mathrm{id}\times\eta_k \\
\mathrm{Bun}_{Q_{k,n}} & \stackrel{\tilde\nu\times\mathrm{id}}{\to} & \widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{Q_k} \\
\downarrow r_k & & \downarrow \\
\mathrm{Bun}_{P(G_{n-k})} & \stackrel{\tilde\nu}{\to} & \widetilde{\mathrm{Bun}}_{G_{n-k}},
\end{array}$$

where the middle square is cartesian. So,

$$(\tilde\nu \times \mathrm{id})^*(\mathrm{id}\times\eta_k)_!\tilde\nu_k^* \mathrm{Aut} \widetilde{\to} (\eta_{k,n})_!\nu_{k,n}^*\tilde\nu^* \mathrm{Aut}$$

Let ${}^0\mathrm{Bun}_{Q_{k,n}} \subset \mathrm{Bun}_{Q_{k,n}}$ be the open substack given by three conditions: $\mathrm{H}^0(X, \mathrm{Sym}^2 L_1) = 0$, $\mathrm{H}^0(X, L_1 \otimes L/L_1) = 0$, and $\mathrm{H}^0(X, \mathrm{Sym}^2(L/L_1)) = 0$. As in Lemma 6, one checks that

$$ {}^0\mathrm{Bun}_{Q_{k,n}} \stackrel{\tilde{\nu}\times\mathrm{id}}{\to} \widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} {}^0\mathrm{Bun}_{Q_k} \tag{21} $$

is smooth and surjective. Since $\eta_{k,n}^{-1}({}^0\mathrm{Bun}_{Q_{k,n}}) \subset \nu_{k,n}^{-1}({}^0\mathrm{Bun}_P)$, from Propositions 7 and 8 we get

$$ (\tilde{\nu} \times \mathrm{id})^*(\mathrm{id} \times \eta_k)_! \tilde{\nu}_k^* \mathrm{Aut} \,\widetilde{\to}\, r_k^* S_{P(G_{n-k}),\psi} \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_G - \dim \mathrm{Bun}_P + a} \tag{22} $$

over ${}^0\mathrm{Bun}_{Q_{k,n}}$. The restriction of r_k to ${}^0\mathrm{Bun}_{Q_{k,n}}$ factors as

$$ {}^0\mathrm{Bun}_{Q_{k,n}} \stackrel{r_k}{\to} {}^0\mathrm{Bun}_{P(G_{n-k})} \hookrightarrow \mathrm{Bun}_{P(G_{n-k})} $$

So, by Proposition 7 applied to G_{n-k}, the RHS of (22) identifies with

$$ (\tilde{\nu} \times \mathrm{id})^*(\mathrm{Aut} \boxtimes \bar{\mathbb{Q}}_\ell) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes d_G - \dim \mathrm{Bun}_P + a + \dim \mathrm{Bun}_{P(G_{n-k})} - d_{G_{n-k}}} $$

We have $b(L_1) = d_G - \dim \mathrm{Bun}_P + a + \dim \mathrm{Bun}_{P(G_{n-k})} - d_{G_{n-k}}$. Since $\mathrm{Bun}_{Q_k} \to \mathrm{Bun}_{G_{n-k}}$ is smooth, $\mathrm{Aut} \boxtimes \bar{\mathbb{Q}}_\ell$ is a shifted perverse sheaf on $\widetilde{\mathrm{Bun}}_{G_{n-k}} \times_{\mathrm{Bun}_{G_{n-k}}} \mathrm{Bun}_{Q_k}$.

Since the restriction of the map (21) to each connected component of ${}^0\mathrm{Bun}_{Q_{k,n}}$ has connected fibres, we get the desired isomorphism.

The second assertion follows from Remark 3 combined with Proposition 8. $\square$

7. Towards geometric θ-lifting

This section is not used in the proofs and may be skipped. Let $\tau_{n,m} : \mathrm{Bun}_{G_n} \times \mathrm{Bun}_{\mathrm{S}\mathbb{O}_m} \to \mathrm{Bun}_{G_{nm}}$ be the following map. Given $\mathrm{S}\mathbb{O}_m$-torsor $\mathcal{F}_W$, let W denote the vector bundle induced from it via the standard representation of $\mathrm{S}\mathbb{O}_m$. Given in addition $M \in \mathrm{Bun}_{G_n}$ we get naturally a symplectic form $\wedge^2(M \otimes W) \to \Omega$. The map $\tau_{n,m}$ sends (M, W) to $M \otimes W$.

Let $\mathcal{A}_{\mathrm{S}\mathbb{O}_m}$ denote the (naturally $\mathbb{Z}/2\mathbb{Z}$-graded) line bundle on $\mathrm{Bun}_{\mathrm{S}\mathbb{O}_m}$, whose fibre at $\mathcal{F}_W$ is $\det \mathrm{R}\Gamma(X, W)$. Write $\mathcal{A}_{G_n}$ to express the dependence on n of the determinant of cohomology on Bun_{G_n}.

Lemma 8. *For $m \geq 3$ we have a $\mathbb{Z}/2\mathbb{Z}$-graded canonical isomorphism over* $\mathrm{Bun}_{G_n} \times \mathrm{Bun}_{\mathrm{S}\mathbb{O}_m}$

$$ \tau_{n,m}^* \mathcal{A}_{G_{nm}} \,\widetilde{\to}\, (\mathcal{A}_{G_n}^m \boxtimes \mathcal{A}_{\mathrm{S}\mathbb{O}_m}^{2n}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{\otimes -2nm} $$

Proof

Step 1. Let us show that for any $M \in \mathrm{Bun}_{G_n}$, $V \in \mathrm{Bun}_{\mathrm{SL}_2}$ we have canonically

$$ \det \mathrm{R}\Gamma(X, M \otimes V) \,\widetilde{\to}\, \det \mathrm{R}\Gamma(X, M)^2 \otimes \det \mathrm{R}\Gamma(X, V)^{2n} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{-4n} $$

Indeed, for $V=\mathcal{O}^2$ we have $\det \mathrm{R}\Gamma(X, M\otimes V)\widetilde{\to} \det\mathrm{R}\Gamma(X,M)^2$. Further, for $M=\mathcal{O}^n\oplus\Omega^n$

$$\det \mathrm{R}\Gamma(X, M\otimes V)\widetilde{\to}\det\mathrm{R}\Gamma(X,V)^n\otimes\det\mathrm{R}\Gamma(X,V\otimes\Omega)^n\widetilde{\to}\det\mathrm{R}\Gamma(X,V)^{2n}$$

Since $\mathrm{H}^0(\mathrm{Bun}_{G_n},\mathcal{O})=\mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL}_2},\mathcal{O})=k$, the assertion follows.

Step 2. Let $\mathcal{F}^0_W$ be the trivial $\mathbb{SO}_m$-torsor on X. Restricting $\tau^*_{n,m}\mathcal{A}_{G_{nm}}$ under $\mathrm{Bun}_{G_n}\stackrel{\mathrm{id}\times\mathcal{F}^0_W}{\to}\mathrm{Bun}_{G_n}\times\mathrm{Bun}_{\mathbb{SO}_m}$, we get $\mathcal{A}^m_{G_n}$ canonically.

For $a\in\mathbb{Z}/2\mathbb{Z}$ denote by $\mathrm{Bun}^a_{\mathbb{SO}_m}$ the corresponding connected component of $\mathrm{Bun}_{\mathbb{SO}_m}$. Let $\mathcal{F}^0_{G_n}$ be the G_n-bundle $\mathcal{O}^n\oplus\Omega^n$ on X. The restriction of $\tau^*_{n,m}\mathcal{A}_{G_{nm}}$ under $\mathcal{F}^0_{G_n}\times\mathrm{id}:\mathrm{Bun}_{\mathbb{SO}_m}\to\mathrm{Bun}_{G_n}\times\mathrm{Bun}_{\mathbb{SO}_m}$ is $\mathcal{A}^{2n}_{\mathbb{SO}_m}$ canonically. This yields the desired isomorphism over $\mathrm{Bun}_{G_n}\times\mathrm{Bun}^0_{\mathbb{SO}_m}$.

If $\mathcal{E}$ is a line bundle on X of odd degree then $W=\mathcal{E}\oplus\mathcal{E}^*\oplus\mathcal{O}^{m-2}\in\mathrm{Bun}^1_{\mathbb{SO}_m}$. For this W we get

$$\det\mathrm{R}\Gamma(X,M\otimes W)\widetilde{\to}\det\mathrm{R}\Gamma(M\otimes(\mathcal{E}\oplus\mathcal{E}^*))\otimes\det\mathrm{R}\Gamma(X,M)^{m-2}$$

By Step 1,

$$\det\mathrm{R}\Gamma(M\otimes(\mathcal{E}\oplus\mathcal{E}^*))\widetilde{\to}\det\mathrm{R}\Gamma(X,M)^2\otimes\det\mathrm{R}\Gamma(X,\mathcal{E}\oplus\mathcal{E}^*)^{2n}\otimes\det\mathrm{R}\Gamma(X,\mathcal{O})^{-4n}$$

The desired isomorphism over $\mathrm{Bun}_{G_n}\times\mathrm{Bun}^1_{\mathbb{SO}_m}$ follows. $\square$

By the lemma combined with 3.1.2, for m even there is a canonical map

$$\tilde\tau_{n,m}:\mathrm{Bun}_{G_n}\times\mathrm{Bun}_{\mathbb{SO}_m}\to\widetilde{\mathrm{Bun}}_{G_{nm}}$$

extending $\tau_{n,m}$. For m odd there is a canonical map

$$\tilde\tau_{n,m}:\widetilde{\mathrm{Bun}}_{G_n}\times\mathrm{Bun}_{\mathbb{SO}_m}\to\widetilde{\mathrm{Bun}}_{G_{nm}}$$

extending $\tau_{n,m}$.

The complex $\tilde\tau^*_{n,m}\mathrm{Aut}$ viewed as a kernel of intergal operators gives rise to a pair of functors between the categories $\mathrm{D}(\widetilde{\mathrm{Bun}}_{G_n})$ and $\mathrm{D}(\mathrm{Bun}_{\mathbb{SO}_m})$ (for m even one may replace $\widetilde{\mathrm{Bun}}_{G_n}$ by Bun_{G_n}). These functors are the geometric counterpart of the classical theta-lifting (in the nonramified case) for the dual reductive pair $\mathbb{S}\mathrm{p}_{2n},\mathbb{SO}_m$ (cf., for example, [19], Sect. 8), we will study them separately.

8. Genuine spherical sheaves on $\widetilde{\mathrm{Gr}}_G$

8.1 Let $\mathcal{O}=k[[t]]$ and $K=k((t))$. Let $\Omega_{\mathcal{O}}$ denote the completed module of relative differentials of $\mathcal{O}$ over k. Pick a free $\mathcal{O}$-module M_0 of rank $2n$ with symplectic form $\wedge^2M_0\to\Omega_{\mathcal{O}}$.

In Sect. 8.1-8.2 G will denote the sheaf of automorphisms of M_0 preserving the symplectic form. One associates to G the affine grassmanian Gr_G (cf. [6], p. 172 or [10]), which is an ind-scheme over k, the fpqc quotient $\mathrm{Gr}_G=G(K)/G(\mathcal{O})$. Here $G(\mathcal{O})$ (resp., $G(K)$) is the functor

associating to a k-algebra R the group of automorphisms of $M_{0,R} := M_0 \otimes_{\mathcal{O}} R[[t]]$ (resp., of $M_0 \otimes_{\mathcal{O}} R((t))$) preserving all the structures.

Recall that the Picard group of Gr_G is $\mathbb{Z}$ (cf. [10]), let us introduce the notation for the generator. We have the affine grassmanian $\mathrm{Gr}_{\mathrm{SL}(M_0)}$. Its R-points are projective $R[[t]]$-modules of finite type $M \subset M_0 \otimes_{\mathcal{O}} R((t))$ with

- $t^m M_{0,R} \subset M \subset t^{-m} M_{0,R}$ for some $m >> 0$;
- $\det_{R[[t]]} M = \det_{R[[t]]} M_{0,R}$ as a subspace of $(\det_{R[[t]]} M_{0,R}) \otimes_{R[[t]]} R((t))$

We postpone to Lemma 9 the proof of the fact that $M/t^m M_{0,R}$ is a projective R-module for $m >> 0$. This allows to introduce the line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\mathrm{SL}(M_0)}$ whose fibre at M is

$$\det(M_0 : M) := \det_R(M_0/t^m M_0) \otimes \det_R(M/t^m M_0)^{-1},$$

independent of m such that $t^m M_0 \subset M$. View it as $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero.

The standard representation of G yields a map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{SL}(M_0)}$, and we also write $\mathcal{L}$ for the restriction of this line bundle to Gr_G. Then $\mathcal{L}$ generates the Picard group of Gr_G. Recall that $\mathcal{L}$ is $G(\mathcal{O})$-equivariant on Gr_G. Let $\widetilde{\mathrm{Gr}}_G \to \mathrm{Gr}_G$ denote the μ_2-gerbe of square roots of $\mathcal{L}$. Then $G(\mathcal{O})$ acts on $\widetilde{\mathrm{Gr}}_G$ extending the action on Gr_G (cf. A.3).

Definition 4. Let $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ be the category of $G(\mathcal{O})$-equivariant perverse sheaves on $\widetilde{\mathrm{Gr}}_G$ on which $-1 \in \mu_2$ acts as -1. We call it the category of *genuine spherical sheaves* on $\widetilde{\mathrm{Gr}}_G$.

A θ-characteristic is a free $\mathcal{O}$-module $\mathcal{N}$ of rank 1 together with $\mathcal{N} \otimes_{\mathcal{O}} \mathcal{N} \widetilde{\to} \Omega_{\mathcal{O}}$. A choice of a θ-characteristic yields an isomorphism of group schemes $G(\mathcal{O}) \widetilde{\to} \mathbb{S}\mathrm{p}(M_0 \otimes_{\mathcal{O}} \mathcal{N}^{-1})$ over k. A further choice of a symplectic base in $M_0 \otimes_{\mathcal{O}} \mathcal{N}^{-1}$ over $\mathcal{O}$ identifies $G(\mathcal{O})$ with $\mathbb{S}\mathrm{p}_{2n}(\mathcal{O})$. So, we may view the standard maximal torus and Borel $T \subset B \subset \mathbb{S}\mathrm{p}_{2n} \subset \mathbb{S}\mathrm{p}_{2n}(\mathcal{O})$ as subgroups of $G(\mathcal{O})$. Write Λ^+ for the set of dominant coweights of $\mathbb{S}\mathrm{p}_{2n}$.

We have a stratification of Gr_G by $G(\mathcal{O})$-orbits indexed by Λ^+, write Gr_G^λ for the $G(\mathcal{O})$-orbit passing by $\lambda(t) \in T(K)$ ([6], p. 180). Let $\widetilde{\mathrm{Gr}}_G^\lambda$ be the preimage of Gr_G^λ in $\widetilde{\mathrm{Gr}}_G$.

Proposition 9. *For any $\lambda \in \Lambda^+$ there is a $G(\mathcal{O})$-equivariant trivialization $\widetilde{\mathrm{Gr}}_G^\lambda \widetilde{\to} \mathrm{Gr}_G^\lambda \times B(\mu_2)$, the $G(\mathcal{O})$-action on the RHS being the product of the action on Gr_G^λ and the trivial action on $B(\mu_2)$.*

Proof

Step 1. For $\lambda \in \Lambda^+$ denote by St_λ the stabilizor of $\lambda(t) \in \mathrm{Gr}_G$ in $G(\mathcal{O})$. Let $M_\lambda = \lambda(t) M_0$ and $M' = M_0 + M_\lambda$, $M'' = M_0 \cap M_\lambda$.

The symplectic form $\wedge^2(M_0 \otimes_{\mathcal{O}} K) \to \Omega(K) = \Omega_{\mathcal{O}} \otimes_{\mathcal{O}} K$ induces a map $(M'/M_0) \otimes (M'/M_\lambda) \widetilde{\to} (M_\lambda/M'') \otimes (M_0/M'') \to \Omega(K)/\Omega_{\mathcal{O}}$. Composing further with the residue map, we get a pairing between k-vector spaces M'/M_0 and M'/M_λ. We'll check in Step 2 that the pairing is perfect. So, the fibre of $\mathcal{L}$ at M_λ is

$$\mathcal{L}_{M_\lambda} \widetilde{\to} \det(M_0 : M_\lambda) \widetilde{\to} \frac{\det(M'/M_\lambda)}{\det(M'/M_0)} \widetilde{\to} \det(M'/M_\lambda)^{\otimes 2}$$

The group St_λ acts on $\det(M'/M_\lambda)$ by some character $\chi : \mathrm{St}_\lambda \to \mathbb{G}_m$. So, St_λ acts on $\mathcal{L}_{M_\lambda}$ by χ^2. Let $\mathcal{B}$ be the $G(\mathcal{O})$-equivariant line bundle on Gr_G^λ corresponding to χ. Then we have a $G(\mathcal{O})$-equivariant isomorphism $\mathcal{B}^2 \widetilde{\to} \mathcal{L}\mid_{\mathrm{Gr}_G^\lambda}$, and our assertion follows from Lemma 17.

Step 2. Realize $\mathbb{S}\mathrm{p}_{2n}$ as the subgroup of SL_{2n} preserving the form on k^{2n} given by the matrix

$$\left(\begin{array}{cc} 0 & E_n \\ -E_n & 0 \end{array}\right),$$

where E_n is the identity matrix in SL_n. Let $T \subset \mathbb{S}\mathrm{p}_{2n}$ be the maximal torus of diagonal matrices. A coweight $\lambda = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)$ of T is dominant iff $a_1 \geq \ldots \geq a_n \geq 0$. Pick a trivialization $\mathcal{N} \widetilde{\to} \mathcal{O}$ and a symplectic base e_i in M_0. Then

$$M_\lambda = t^{a_1}\mathcal{O}e_1 \oplus \ldots \oplus t^{a_n}\mathcal{O}e_n \oplus t^{-a_1}\mathcal{O}e_{n+1} \oplus \ldots \oplus t^{-a_n}\mathcal{O}e_{2n}$$

and $M' = \mathcal{O}e_1 \oplus \ldots \oplus \mathcal{O}e_n \oplus t^{-a_1}\mathcal{O}e_{n+1} \oplus \ldots \oplus t^{-a_n}\mathcal{O}e_{2n}$. Since

$$M'/M_0 \widetilde{\to} t^{-a_1}\mathcal{O}e_{n+1} \oplus \ldots \oplus t^{-a_n}\mathcal{O}e_{2n}/\mathcal{O}e_{n+1} \oplus \ldots \oplus \mathcal{O}e_{2n}$$

$$M'/M_\lambda \widetilde{\to} \mathcal{O}e_1 \oplus \ldots \oplus \mathcal{O}e_n/t^{a_1}\mathcal{O}e_1 \oplus \ldots \oplus t^{a_n}\mathcal{O}e_n,$$

the pairing is perfect. □

Let W denote the nontrivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\operatorname{Spec} k \to B(\mu_2)$. For $\lambda \in \Lambda^+$ there is a unique irreducible $G(\mathcal{O})$-equivariant perverse sheaf on $\widetilde{\mathrm{Gr}}_G^\lambda$, on which $-1 \in \mu_2$ acts as -1, namely $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes \dim \mathrm{Gr}_G^\lambda}$. Denote by $\mathcal{A}_\lambda$ its Goresky-MacPherson extension to $\widetilde{\mathrm{Gr}}_G$. By Proposition 9, the irreducible objects of the category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ are exactly $\mathcal{A}_\lambda, \lambda \in \Lambda^+$.

Note that $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is closed under extensions in $\mathrm{P}(\widetilde{\mathrm{Gr}}_G)$ (if $-1 \in \mu_2$ acts as -1 on perverse sheaves K_1, K_2 then it acts as -1 on any extension of K_1 by K_2). Since $\mathbb{D}(\mathcal{A}_\lambda) \widetilde{\to} \mathcal{A}_\lambda$ canonically, $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is preserved by Verdier duality.

Consider the action of the torus $T \subset G(\mathcal{O})$ on Gr_G. The following will be used in Sect. 8.4.

Lemma 9. *i) There is a covering of* Gr_G *by* T*-invariant open ind-schemes* U_i *and* T*-equivariant trivializations* $\mathcal{L}\mid_{U_i} \widetilde{\to} \mathcal{O}_{U_i}$.
ii) For an R*-point* $M \subset M_0 \otimes_{\mathcal{O}} R((t))$ *of* $\mathrm{Gr}_{\mathrm{SL}(M_0)}$ *and* $m >> 0$ *the* R*-module* $M/t^m M_{0,R}$ *is projective.*

Proof i) Pick a trivialization $\mathcal{N} \widetilde{\to} \mathcal{O}$, so that our base of $M_0 \otimes \mathcal{N}^{-1}$ gives rise to a base $\{e_1, \ldots, e_{2n}\}$ of M_0. Consider the corresponding maximal torus T' of $\mathrm{SL}(M_0)$. Set $M^- = Ae_1 \oplus \ldots \oplus Ae_{2n}$ with $A = t^{-1}k[t^{-1}]$. For a coweight $\lambda : \mathbb{G}_m \to T'$ of $\mathrm{SL}(M_0)$ denote by $U_\lambda \subset \mathrm{Gr}_{\mathrm{SL}(M_0)}$ the open locus classifying lattices $M \subset M_0 \otimes_{\mathcal{O}} K$ such that $M \oplus \lambda(t)M^- = M_0 \otimes_{\mathcal{O}} K$. Here $\lambda = (b_1, \ldots, b_{2n})$ with $\sum b_i = 0$ and $\lambda(t)M^- = At^{b_1}e_1 \oplus \ldots \oplus At^{b_{2n}}e_{2n}$.

One checks that the union of U_λ is $\mathrm{Gr}_{\mathrm{SL}(M_0)}$. Clearly, U_λ is T'-invariant. As shown by Faltings ([10], Sect. 2), for each λ there is a trivialization $\mathcal{L}\mid_{U_\lambda} \widetilde{\to} \mathcal{O}_{U_\lambda}$ equivariant under

the stabilizor of $\lambda(t)M^-$ in $\mathrm{SL}(M_0)(K)$. This stabilizor contains T', so the trivializations are T'-equivariant.

Restricting everything under the map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{SL}(M_0)}$ corresponding to the standard representation, one concludes the proof.

ii) (argument due to the unknown referee) Localizing in Zarisky topology of R, pick a coweight λ of $\mathrm{SL}(M_0)$ such that $M \oplus \lambda(t)M_R^- = M_0 \otimes_{\mathcal{O}} R((t))$. Here $M_R^- = A_R e_1 \oplus \ldots \oplus A_R e_{2n}$ and $A_R = t^{-1}R[t^{-1}]$. For $m >> 0$ we get $t^{-m}M_{0,R} = M \oplus U$, where $U = \lambda(t)M_R^- \cap t^{-m}M_{0,R}$, and

$$(M/t^m M_{0,R}) \oplus U \widetilde{\to} t^{-m}M_{0,R}/t^m M_{0,R}$$

□

8.2 The convolution product. Following [17], consider the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \stackrel{p_G\times\mathrm{id}}{\leftarrow} G(K) \times \mathrm{Gr}_G \stackrel{q_G}{\to} G(K) \times_{G(\mathcal{O})} \mathrm{Gr}_G \stackrel{m}{\to} \mathrm{Gr}_G,$$

Here $p_G : G(K) \to \mathrm{Gr}_G$ is the projection, $G(K) \times_{G(\mathcal{O})} \mathrm{Gr}_G$ is the quotient of $G(K) \times \mathrm{Gr}_G$ by $G(\mathcal{O})$, where the action is given by $x(g, hG(\mathcal{O})) = (gx^{-1}, xhG(\mathcal{O}))$ for $x \in G(\mathcal{O})$, and m is the product map.

The map $p_G \times m : G(K) \times_{G(\mathcal{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$ sending $(g, hG(\mathcal{O}))$ to $(gG(\mathcal{O}), ghG(\mathcal{O}))$ is an isomorphism.

We have a canonical isomorphism $q_G^* m^*\mathcal{L} \widetilde{\to} p_G^*\mathcal{L} \boxtimes \mathcal{L}$. Moreover, the above $G(\mathcal{O})$-action on $G(K) \times \mathrm{Gr}_G$ lifts to a $G(\mathcal{O})$-equivariant structure on $p_G^*\mathcal{L} \boxtimes \mathcal{L}$ giving rise to the line bundle $p_G^*\mathcal{L} \tilde{\boxtimes} \mathcal{L}$ on $G(K) \times_{G(\mathcal{O})} \mathrm{Gr}_G$. Thus, $m^*\mathcal{L} \widetilde{\to} p_G^*\mathcal{L} \tilde{\boxtimes} \mathcal{L}$ canonically.

Set $\widetilde{G(K)} = G(K) \times_{\mathrm{Gr}_G} \widetilde{\mathrm{Gr}}_G$. Both actions of $G(\mathcal{O})$ on $G(K)$ by left and right translations extend naturally to actions on $\widetilde{G(K)}$. We'll refer to them again as actions by left and right translations, by abuse of terminology. Under the action on $\widetilde{G(K)}$ by right translations, the projection $\tilde{p}_G : \widetilde{G(K)} \to \widetilde{\mathrm{Gr}}_G$ is a $G(\mathcal{O})$-torsor (cf. A.2).

Taking the tensor product of square roots of $p_G^*\mathcal{L}$ and of $\mathcal{L}$, we get a map $\tilde{m}$ that fits into the diagram

$$\begin{array}{ccc} \widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G & \stackrel{\tilde{m}}{\to} & \widetilde{\mathrm{Gr}}_G \\ \downarrow & & \downarrow \\ G(K) \times \mathrm{Gr}_G & \stackrel{m\circ q_G}{\to} & \mathrm{Gr}_G \end{array}$$

One checks that

$$\tilde{p}_G \times \tilde{m} : \widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G \to \widetilde{\mathrm{Gr}}_G \times \widetilde{\mathrm{Gr}}_G \tag{23}$$

is a $G(\mathcal{O})$-torsor, where $G(\mathcal{O})$ acts on $\widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G$ as the product of the action by right translations on $\widetilde{G(K)}$ with the action on $\widetilde{\mathrm{Gr}}_G$.

Consider the diagram

$$\widetilde{\mathrm{Gr}}_G \times \widetilde{\mathrm{Gr}}_G \stackrel{\tilde{p}_G\times\mathrm{id}}{\leftarrow} \widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G \stackrel{\tilde{p}_G\times\tilde{m}}{\to} \widetilde{\mathrm{Gr}}_G \times \widetilde{\mathrm{Gr}}_G \stackrel{pr_2}{\to} \widetilde{\mathrm{Gr}}_G$$

Definition 5. For $K_1, K_2 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ define the convolution product $K_1 * K_2 \in \mathrm{D}(\widetilde{\mathrm{Gr}}_G)$ by

$$K_1 * K_2 = \mathrm{pr}_{2!} K,$$

where K is a perverse sheaf on $\widetilde{\mathrm{Gr}}_G \times \widetilde{\mathrm{Gr}}_G$ such that $(\tilde{p}_G \times \tilde{m})^* K \widetilde{\to} \tilde{p}_G^* K_1 \boxtimes K_2$. Since (23) is a $G(\mathcal{O})$-torsor and $\tilde{p}_G^* K_1 \boxtimes K_2$ is equivariant under the corresponding $G(\mathcal{O})$-action on $\widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G$, K is defined up to a unique isomorphism (cf. A.2).

For $(a, b) \in \mu_2 \times \mu_2$ the image under $\tilde{p}_G \times \tilde{m}$ of the corresponding 2-automorphism of $\widetilde{G(K)} \times \widetilde{\mathrm{Gr}}_G$ is the 2-automorphism (a, ab) of $\widetilde{\mathrm{Gr}}_G \times \widetilde{\mathrm{Gr}}_G$. So, by Lemma 16, K descends to a perverse sheaf K' on $\mathrm{Gr}_G \tilde{\times} \widetilde{\mathrm{Gr}}_G$ (such K' is defined up to a unique isomorphism). Since $\mathrm{R}\Gamma_c(B(\mu_2), \bar{\mathbb{Q}}_\ell) = \bar{\mathbb{Q}}_\ell$, we see that $K_1 * K_2 \widetilde{\to} \mathrm{pr}_{2!} K'$, where $\mathrm{pr}_2 : \mathrm{Gr}_G \tilde{\times} \widetilde{\mathrm{Gr}}_G \to \widetilde{\mathrm{Gr}}_G$ is the projection. Moreover, $-1 \in \mu_2$ acts on $K_1 * K_2$ as -1.

Proposition 10. *For $K_1, K_2 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ we have $K_1 * K_2 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$.*

Proof Following [17], stratify $\mathrm{Gr}_G \tilde{\times} \widetilde{\mathrm{Gr}}_G$ by locally closed substacks $\widetilde{\mathrm{Gr}}_G^{\lambda,\mu}$, $\lambda, \mu \in \Lambda^+$, where $\widetilde{\mathrm{Gr}}_G^{\lambda,\mu}$ is the preimage of $(p_G \times m)(p_G^{-1}(\mathrm{Gr}_G^\lambda) \times_{G(\mathcal{O})} \mathrm{Gr}_G^\mu)$ under $\mathrm{Gr}_G \tilde{\times} \widetilde{\mathrm{Gr}}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G$.

Stratify also $\widetilde{\mathrm{Gr}}_G$ by $\widetilde{\mathrm{Gr}}_G^\lambda$, $\lambda \in \Lambda^+$. By Lemma 4.3 of *loc.cit.*, $\mathrm{pr}_2 : \mathrm{Gr}_G \tilde{\times} \widetilde{\mathrm{Gr}}_G \to \widetilde{\mathrm{Gr}}_G$ is stratified semi-small map. Our assertion follows from Lemma 4.2 of *loc.cit.* $\square$

In a similar way one defines a convolution product $K_1 * K_2 * K_3$ of three sheaves $K_1, K_2, K_3 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. Moreover, $(K_1 * K_2) * K_3 \widetilde{\to} K_1 * K_2 * K_3 \widetilde{\to} K_1 * (K_2 * K_3)$ canonically, and $\mathcal{A}_0$ is a unit object. So, $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is an associative tensor category (a category with tensor functor and an associativity constraint).

Observe that for each $\lambda \in \Lambda^+$ the $G(\mathcal{O})$-orbit Gr_G^λ is even-dimensional.

Proposition 11. *1) For $\lambda \in \Lambda^+$ the odd cohomology sheaves of $\mathcal{A}_\lambda$ (with respect to the usual t-structure) vanish.*
2) The category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is semisimple.

Proof 1a) Given $\lambda_1, \ldots, \lambda_r \in \Lambda^+$, consider the convolution diagram

$$m : \mathrm{Conv}^{\lambda_1, \ldots, \lambda_r} \to \overline{\mathrm{Gr}}_G^{\lambda_1 + \ldots + \lambda_r},$$

where we have set $\mathrm{Conv}^{\lambda_1, \ldots, \lambda_r} = \mathrm{Gr}_G^{\lambda_1} \tilde{\times} \ldots \tilde{\times} \mathrm{Gr}_G^{\lambda_r}$. Let $\widetilde{\mathrm{Conv}}^{\lambda_1, \ldots, \lambda_r}$ be the restriction of the gerbe $\widetilde{\mathrm{Gr}}_G$ under the above map m. The canonical section $s : \mathrm{Gr}_G^{\lambda_1 + \ldots + \lambda_r} \to \widetilde{\mathrm{Gr}}_G^{\lambda_1 + \ldots + \lambda_r}$ yields a section $m^{-1}(s)$ of the gerbe $\widetilde{\mathrm{Conv}}^{\lambda_1, \ldots, \lambda_r}$ over $m^{-1}(\mathrm{Gr}_G^{\lambda_1 + \ldots + \lambda_r})$. One checks that this section extends to a section $\mathrm{Conv}^{\lambda_1, \ldots, \lambda_r} \to \widetilde{\mathrm{Conv}}^{\lambda_1, \ldots, \lambda_r}$.

1b) We adopt Gaitsgory's proof of a theorem of Lusztig to our situation ([11], A.7). Namely, let $\mathcal{F}l$ denote the affine flag variety. This is the ind-scheme classifying a G-bundle $\mathcal{F}_G$ on $\mathrm{Spec}\, \mathcal{O}$ with trivialization $\mathcal{F}_G \widetilde{\to} \mathcal{F}_G^0 \mid_{\mathrm{Spec}\, K}$ and a reduction of $\mathcal{F}_G \mid_{\mathrm{Spec}\, \mathcal{O}/(t)}$ to the Borel subgroup B.

Let $\widetilde{\mathcal{F}l}$ denote the restriction of the gerbe $\widetilde{\mathrm{Gr}}_G$ under the (smooth) projection $\mathcal{F}l \to \mathrm{Gr}_G$. Let $I \subset G(\mathcal{O})$ be the Iwahory subgroup. For an element w of the affine Weil group of G, let $\mathcal{F}l^w$ denote the corresponding I-orbit on $\mathcal{F}l$. Set $\widetilde{\mathcal{F}l}^w = \mathcal{F}l^w \times_{\mathcal{F}l} \widetilde{\mathcal{F}l}$.

Let $\mu \in \Lambda^+$ be such that the projection $\mathcal{F}l^w \to \mathrm{Gr}_G$ factors through Gr_G^μ. The canonical section $\mathrm{Gr}_G^\mu \to \widetilde{\mathrm{Gr}}_G^\mu$ yields a section $s : \mathcal{F}l^w \to \widetilde{\mathcal{F}l}^w$ of the gerbe $\widetilde{\mathcal{F}l}^w \to \mathcal{F}l^w$. Let $\mathcal{A}_w$ denote the irreducible perverse sheaf on the closure of $\widetilde{\mathcal{F}l}^w$ on which $-1 \in \mu_2$ acts as -1 and whose restriction under s is $\mathrm{IC}_{\mathcal{F}l^w}$. It suffices to show the parity vanishing for stalks of $\mathcal{A}_w$.

Let $w = s_1 \cdot \ldots \cdot s_r$ be a reduced decomposition of w into a product of simple reflections. Denote by $p : \mathrm{Conv}_{\mathcal{F}l}^{s_1,\ldots,s_r} \to \overline{\mathcal{F}l}^w$ the Bott-Samelson resolution (*loc.cit.* or [10], Sect. 3, where it is referred to as Demazure resolution). Let $\widetilde{\mathrm{Conv}}_{\mathcal{F}l}^{s_1,\ldots,s_r}$ be the restriction of our gerbe to $\mathrm{Conv}_{\mathcal{F}l}^{s_1,\ldots,s_r}$. By 1a), the section

$$p^{-1}(\mathcal{F}l^w) \to p^{-1}(\widetilde{\mathcal{F}l}^w)$$

induced by s extends to a global section $\mathrm{Conv}_{\mathcal{F}l}^{s_1,\ldots,s_r} \to \widetilde{\mathrm{Conv}}_{\mathcal{F}l}^{s_1,\ldots,s_r}$. The desired assertion follows, because the fibres of p have cohomology with compact support in even degrees only ([11], A.7).

2) Follows from 1) as in ([6], 5.3.3). This uses the fact that each Gr_G^λ has cohomology only in even degrees (5.3.2 of *loc.cit.*). $\square$

Remark 5. The group of automorphisms of the k-algebra $\mathcal{O}$ is naturally the group of k-points of a (reduced) affine group scheme $\mathrm{Aut}^0 \mathcal{O}$ over k ([6], 2.6.5). Assume that $M_0 = \mathcal{O}^n \oplus \Omega^n_{\mathcal{O}}$ with standard symplectic form. Then $\mathrm{Aut}^0 \mathcal{O}$ acts on M_0 and, hence, on Gr_G. Moreover, $\mathcal{L}$ is naturally equivariant under this action. It follows that $\mathrm{Aut}^0 \mathcal{O}$ acts on $\widetilde{\mathrm{Gr}}_G$. Proposition 9 then can be strengthened saying that the gerbe $\widetilde{\mathrm{Gr}}_G^\lambda \to \mathrm{Gr}_G^\lambda$ admits a $G(\mathcal{O}) \rtimes \mathrm{Aut}^0 \mathcal{O}$-equivariant trivialization.

We also see that each $\mathcal{A}_\lambda$ is $G(\mathcal{O}) \rtimes \mathrm{Aut}^0 \mathcal{O}$-equivariant (this property is true for the constant sheaf over Gr_G^λ and is preserved under intermediate extension). By Proposition 11, each $K \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is $\mathrm{Aut}^0 \mathcal{O}$-equivariant. Moreover, such equivariant structure is unique (because the stabilizer of a point is connected) and compatible with any morphism in $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$.

8.3 THE FUSION PRODUCT Following [17], we will show that the convolution product defined above can be interpreted as a 'fusion' product, thus leading to a commutativity constraint on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. The original idea of this interpretation for spherical sheaves on Gr_G is due to V. Drinfeld.

Let G denote the sheaf of groups on X introduced in Sect. 3.2. For $x \in X(k)$ write $\mathcal{O}_x$ for the completed local ring at x and K_x for its fraction field. Write $\mathrm{Gr}_{G,x} = G(K_x)/G(\mathcal{O}_x)$ for the corresponding version of the affine grassmanian.

Write $\mathcal{F}_G^0$ for the 'trivial' G-torsor on X given by $M_0 = \mathcal{O}_X^n \oplus \Omega^n$ with standard symplectic form $\wedge^2 M_0 \to \Omega$.

For a k-algebra R write $X_R = X \times \mathrm{Spec}\, R$ and $X_R^* = (X - x) \times \mathrm{Spec}\, R$. By [1, 2], $\mathrm{Gr}_{G,x}$ is the functor on the category of k-algebras sending R to the set of isomorphism classes of $\{\mathcal{F}_G, \nu\}$, where $\mathcal{F}_G$ is a G-torsor on X_R and $\nu : \mathcal{F}_G \widetilde{\to} \mathcal{F}_G^0 \mid_{X_R^*}$ is a trivialization outside x.

Let M denote the vector bundle on X induced from $\mathcal{F}_G$ via the standard representation of G. Set $M_x = M \otimes \mathcal{O}_x$ and $M_{0,x} = M_0 \otimes \mathcal{O}_x$. Then $M_x \subset M_{0,x} \otimes_{\mathcal{O}_x} K_x$ is a sublattice, and we continue to denote by $\mathcal{L}$ the line bundle on $\mathrm{Gr}_{G,x}$ with fibre $\det(M_{0,x} : M_x)$. Then $\widetilde{\mathrm{Gr}}_{G,x}$ and $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ are defined as in Sect. 8.1.

Write Gr_{G,X^d} for the functor associating to a k-algebra R the set

$$\{(x_1, \ldots, x_d) \in X^d(R),\ \text{a } G-\text{torsor } \mathcal{F}_G \text{ on } X_R,\ \mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{X_R - \cup x_i}\}$$

Here $x_i \in X(R)$ are thought of as subschemes in X_R by taking their graphs.

Let G_{X^d} denote the functor sending a k-algebra R to the set $\{(x_1, \ldots, x_d) \in X^d(R), \mu)$, where μ is an automorphism of $\mathcal{F}^0_G$ restricted to the formal neighborhood $\widehat{X}_{R,D}$ of $D = x_1 \cup \ldots \cup x_d$ in X_R. So, G_{X^d} is a group scheme over X^d, whose fibre over $(x_1, \ldots, x_d)$ is $\prod_i G(\mathcal{O}_{y_i})$ with $\{y_1, \ldots, y_s\} = \{x_1, \ldots, x_d\}$ and y_i pairwise distinct.

Let $\mathcal{L}$ be the line bundle on Gr_{G,X^n} whose fibre is $\det \mathrm{R}\Gamma(X, M_0) \otimes \det \mathrm{R}\Gamma(X, M)^{-1}$, where M is the vector bundle on X induced from $\mathcal{F}_G$ via the standard representation of G.

Lemma 10. *For a k-point $(x_1, \ldots, x_d, \mathcal{F}_G)$ of Gr_{G,X^d} let $\{y_1, \ldots, y_s\} = \{x_1, \ldots, x_d\}$ with y_i pairwise distinct. The fibre of $\mathcal{L}$ at this k-point is canonically isomorphic (as $\mathbb{Z}/2\mathbb{Z}$-graded) to*

$$\otimes_{i=1}^s \det(M_{0,y_i} : M_{y_i}) \qquad \square$$

One checks that the natural action of G_{X^d} on Gr_{G,X^d} lifts to a G_{X^d}-equivariant structure on $\mathcal{L}$. We have $\widetilde{\mathrm{Gr}}_{G,X^d}$ and $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,X^d})$ defined as above.

8.3.1 Consider the diagram of stacks over X^2, where the left and right square is cartesian

$$\begin{array}{ccccccc}
\widetilde{\mathrm{Gr}}_{G,X} \times \widetilde{\mathrm{Gr}}_{G,X} & \stackrel{\tilde p_{G,X}}{\leftarrow} & \widetilde{C}_{G,X} & \stackrel{\tilde q_{G,X}}{\to} & \widetilde{\mathrm{Conv}}_{G,X} & \stackrel{\tilde m_X}{\to} & \widetilde{\mathrm{Gr}}_{G,X^2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} & \stackrel{p_{G,X}}{\leftarrow} & C_{G,X} & \stackrel{q_{G,X}}{\to} & \mathrm{Conv}_{G,X} & \stackrel{m_X}{\to} & \mathrm{Gr}_{G,X^2}
\end{array}$$

Here the low row is the usual convolution diagram [17], (5.2). Namely, $C_{G,X}$ is the ind-scheme classifying collections:

$$\left\{ \begin{array}{l} x_1, x_2 \in X,\ G-\text{torsors } \mathcal{F}^1_G, \mathcal{F}^2_G \text{ on } X \text{ with trivializations } \nu_i : \mathcal{F}^i_G \widetilde{\to} \mathcal{F}^0_G \mid_{X - x_i}, \\ \mu_1 : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}^0_G \mid_{\widehat{X}_{x_2}}, \end{array} \right. \tag{24}$$

where $\widehat{X}_{x_2}$ is the formal neighborhood of x_2 in X. The map $p_{G,X}$ forgets μ_1.

The ind-scheme $\mathrm{Conv}_{G,X}$ classifies collections:

$$\left\{ \begin{array}{l} x_1, x_2 \in X,\ G-\text{torsors } \mathcal{F}^1_G, \mathcal{F}_G \text{ on } X, \\ \text{isomorphisms } \nu_1 : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}^0_G \mid_{X - x_1}, \text{ and } \eta : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}_G \mid_{X - x_2} \end{array} \right. \tag{25}$$

The map m_X sends this collection to $(x_1, x_2, \mathcal{F}_G)$ together with the trivialization $\eta \circ \nu_1^{-1} : \mathcal{F}^0_G \widetilde{\to} \mathcal{F}_G \mid_{X - x_1 - x_2}$.

The map $q_{G,X}$ sends (24) to the collection (25), where $\mathcal{F}_G$ is obtained by gluing $\mathcal{F}^1_G$ on $X-x_2$ and $\mathcal{F}^2_G$ on $\widehat{X}_{x_2}$ using their identification over $(X-x_2)\cap\widehat{X}_{x_2}$ via $\nu_2^{-1}\circ\mu_1$.

The canonical isomorphism

$$q^*_{G,X}m^*_X\mathcal{L}\,\widetilde{\to}\,p^*_{G,X}(\mathcal{L}\boxtimes\mathcal{L})$$

allows to define $\tilde q_{G,X}$ as follows. Write M_i (resp., M) for the vector bundle induced from $\mathcal{F}^i_G$ (resp., $\mathcal{F}_G$) via the standard representation of G.

A point of $\tilde C_{G,X}$ is given by (24) together with 1-dimensional vector spaces $\mathcal{B}_1,\mathcal{B}_2$ and $\mathcal{B}_i^2\,\widetilde{\to}\,\mathcal{L}_{\mathcal{F}^i_G}$. By Lemma 10, $\mathcal{L}_{\mathcal{F}^i_G}\,\widetilde{\to}\,\det(M_{0,x_i}:\det M_{i,x_i})$.

A point of $\widetilde{\operatorname{Conv}}_{G,X}$ is given by (25) together with 1-dimensional vector space $\mathcal{B}$ and $\mathcal{B}^2\,\widetilde{\to}\,\mathcal{L}_{\mathcal{F}_G}$. We have

$$\mathcal{L}_{\mathcal{F}_G}\,\widetilde{\to}\,\frac{\det\mathrm{R}\Gamma(X,M_0)}{\det\mathrm{R}\Gamma(X,M_1)}\otimes\frac{\det\mathrm{R}\Gamma(X,M_1)}{\det\mathrm{R}\Gamma(X,M)}\,\widetilde{\to}\,\det(M_{0,x_1}:M_{1,x_1})\otimes\det(M_{1,x_2}:M_{x_2})\,\widetilde{\to}\,\mathcal{L}_{\mathcal{F}^1_G}\otimes\mathcal{L}_{\mathcal{F}^2_G},$$

the last isomorphism being given by $\mu_1:\det(M_{1,x_2})\,\widetilde{\to}\,\det(M_{0,x_2})$ and $M_{x_2}\,\widetilde{\to}\,M_{2,x_2}$. Define $\tilde q_{G,X}$ by setting $\mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2$.

As in Sect. 8.2 one checks that for $K_1,K_2\in\operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})$ there is a (defined up to a unique isomoprhism) perverse sheaf K_{12} on $\widetilde{\operatorname{Conv}}_{G,X}$ with $\tilde q^*_{G,X}K_{12}\,\widetilde{\to}\,\tilde p^*_{G,X}(K_1\boxtimes K_2)$. Moreover, $-1\in\mu_2$ acts on K_{12} as -1. We then let

$$K_1*_X K_2=\tilde m_{X!}K_{12}$$

Let $U\subset X^2$ be the complement to the diagonal. Let $j:\widetilde{\operatorname{Gr}}_{G,X^2}(U)\hookrightarrow\widetilde{\operatorname{Gr}}_{G,X^2}$ be the preimage of U. Recall that m_X is stratified small, an isomorphism over the preimage of U ([17]). So, the same holds for the representable map $\tilde m_X$. Thus, $K_1*_X K_2$ is a perverse sheaf, the Goresky-MacPherson from $\widetilde{\operatorname{Gr}}_{G,X^2}(U)$. Besides, $-1\in\mu_2$ acts on it as -1. Moreover, $K_1*_X K_2\in\operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X^2})$, because G_{X^2}-equivariance is clear over $\widetilde{\operatorname{Gr}}_{G,X^2}(U)$ and is preserved under the intermediate extension.

Recall the group ind-scheme $\operatorname{Aut}^0\mathcal{O}$ (cf. Remark 5). Let $\hat X\to X$ be the $\operatorname{Aut}^0\mathcal{O}$-torsor whose fibre is the set of all trivializations $\mathcal{O}_x\,\widetilde{\to}\,\mathcal{O}$. It is known that $\operatorname{Gr}_{G,X}\,\widetilde{\to}\,\hat X\times_{\operatorname{Aut}^0\mathcal{O}}\operatorname{Gr}_G$ ([6], 5.3.11). The line bundle $\mathcal{L}$ on $\operatorname{Gr}_{G,X}$ identifies with the descent of the $\operatorname{Aut}^0\mathcal{O}$-equivariant line bundle $\mathcal{O}\boxtimes\mathcal{L}$ under $\hat X\times\operatorname{Gr}_G\to\operatorname{Gr}_{G,X}$. Since any $K\in\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$ is $\operatorname{Aut}^0\mathcal{O}$-equivariant, we have a natural (fully faithful) functor

$$\tau^0:\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)\to\operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})[-1] \tag{26}$$

Let $\operatorname{glob}:\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)\to\operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})$ denote the functor $\operatorname{glob}=\tau^0[1]$.

Now define the commutativity constraint following [17]. Let $i:\widetilde{\operatorname{Gr}}_{G,X}\to\widetilde{\operatorname{Gr}}_{G,X^2}$ be the preimage of the diagonal in X^2. For $F_1,F_2\in\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$ letting $K_i=\tau^0F_i$ define

$$K_{12}\mid_U:=K_{12}\mid_{\widetilde{\operatorname{Gr}}_{G,X^2}(U)}$$

as above (but now it is placed in perverse degree 2). We get

$$K_1 *_X K_2 \widetilde{\to} j_{!*}(K_{12} \mid_U) \tag{27}$$
$$\tau^0(F_1 * F_2) \widetilde{\to} i^*(K_1 *_X K_2) \tag{28}$$

So, the involution σ of $\widetilde{\mathrm{Gr}}_{G,X^2}$ interchanging x_i yields

$$\tau^0(F_1 * F_2) \widetilde{\to} i^* j_{!*}(K_{12} \mid_U) \widetilde{\to} i^* j_{!*}(K_{21} \mid_U) \widetilde{\to} \tau^0(F_2 * F_1),$$

because $\sigma^*(K_{12} \mid_U) \widetilde{\to} K_{21} \mid_U$. (We used the functor τ^0 instead of glob to avoid the signs ambiguity in the commutativity constraints).

To show that the associativity and commutativity constraints are compatible, argue as in ([6], 5.3.13-5.3.17). Namely, one defines for a non-empty finite set I a category $\otimes_I \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ and for any surjection $h : I \to I'$ a functor $*_h : \otimes_I \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G) \to \otimes_{I'} \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. They are compatible in the sense of (*loc.cit.*, (266)). Thus, $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ is a tensor category.

Remark 6. Fix $x \in X(k)$. Consider the Hecke stack ${}_x\mathcal{H}_G$ classifying two G-bundles $\mathcal{F}_G, \mathcal{F}'_G$ on X together with an isomorphism $\mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{X-x}$. Let p (resp., p') be the projection ${}_x\mathcal{H}_G \to \mathrm{Bun}_G$ sending the above collection to $\mathcal{F}_G$ (resp., $\mathcal{F}'_G$). Write Bun_G^x for the stack classifying a G-torsor $\mathcal{F}_G$ on X together with a trivialization $\mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{D_x}$ over the formal disk D_x around x.

Let γ (resp., γ') be the isomorphism $\mathrm{Bun}_G^x \times_{G(\mathcal{O}_x)} \mathrm{Gr}_{G,x} \widetilde{\to} {}_x\mathcal{H}_G$ such that the projection to the first term corresponds to p (resp., to p'). Write M (resp., M') for the vector bundle corresponding to $\mathcal{F}_G$ (resp., to $\mathcal{F}'_G$) via the standard representation of G. Write $\mathcal{L}$ for the ($\mathbb{Z}/2\mathbb{Z}$-graded) line bundle on ${}_x\mathcal{H}_G$ with fibre $\det \mathrm{R}\Gamma(X, M) \otimes \det \mathrm{R}\Gamma(X, M')^{-1}$. Let ${}_x\tilde{\mathcal{H}}_G$ be the gerbe of square roots of $\mathcal{L}$. Both γ and γ' extend to $G(\mathcal{O}_x)$-torsors

$$\tilde{\gamma}, \tilde{\gamma}' : \mathrm{Bun}_G^x \times \widetilde{\mathrm{Gr}}_{G,x} \to {}_x\tilde{\mathcal{H}}_G$$

For $\mathcal{S} \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ denote by $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{S}$ (resp., by $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes}' \mathcal{S}$) the twisted tensor product viewed as a perverse sheaf on ${}_x\tilde{\mathcal{H}}_G$ via $\tilde{\gamma}$ (resp., $\tilde{\gamma}'$). Given $\mathcal{S} \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ there is a (defined up to a unique isomorphism) $\mathcal{T} \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ equipped with an isomorphism $\bar{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{S} \widetilde{\to} \bar{\mathbb{Q}}_\ell \tilde{\boxtimes}' \mathcal{T}$. This defines a covariant involution functor $\star$ on the category $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ By Remark 5, we may view $\star$ as an involution functor on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ independently of a choice of a trivialization $\mathcal{O}_x \widetilde{\to} \mathcal{O}$.

In the same way as for usual spherical sheaves on Gr_G, one checks that for $K_1, K_2, K_3 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ we have canonically $R\mathrm{Hom}(K_1 * K_2, K_3) \widetilde{\to} R\mathrm{Hom}(K_1, K_3 * \mathbb{D}(\star K_2))$. So, $K_3 * \mathbb{D}(\star K_2)$ represents the internal $\mathcal{H}om(K_2, K_3)$ in the sense of the tensor structure on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. Besides, $\star(K_1 * K_2) \widetilde{\to} (\star K_2) * (\star K_1)$ canonically. We also have $\mathbb{D}(\star \mathcal{A}_\lambda) \widetilde{\to} \star \mathcal{A}_\lambda \widetilde{\to} \mathcal{A}_\lambda$ for each $\lambda \in \Lambda_+$.

8.4 FUNCTORS F^θ. Let $P \subset G$ denote the Siegel parabolic preserving $\mathcal{O}_X^n \subset \mathcal{O}_X^n \oplus \Omega^n$. Write Q for the Levi quotient, so $Q \widetilde{\to} \mathrm{GL}_n$ canonically. Let $\check{\Lambda}_{G,P}$ denote the lattice of characters of $P/[P,P] = Q/[Q,Q]$ and $\Lambda_{G,P}$ the dual lattice. Let $\check{\omega}_n \in \check{\Lambda}_{G,P}$ denote the fundamental weight of G corresponding to the unique simple coroot which is not a coroot of Q. So, $\check{\omega}_n$ is the highest

weight of an irreducible subrepresentation in $\wedge^n M$, where M is the standard representation of G. Then $\check{\omega}_n$ is a free generator of $\check{\Lambda}_{G,P}$.

The connected components of $\mathrm{Gr}_{Q,x}$ are indexed by $\Lambda_{G,P}$, the component $\mathrm{Gr}^\theta_{Q,x}$ classifies $(L\in \mathrm{Bun}_n, \nu: L\,\widetilde{\to}\,\mathcal{O}^n\mid_{X-x})$ such that $\deg L=-\langle\theta,\check{\omega}_n\rangle$. The reduced part $\mathrm{Gr}^\theta_{Q,x,red}\hookrightarrow \mathrm{Gr}^\theta_{Q,x}$ is the ind-scheme classifying $(L\in \mathrm{Bun}_n, \nu: L\,\widetilde{\to}\,\mathcal{O}^n\mid_{X-x})$ that induce an isomorphism

$$\det L\,\widetilde{\to}\,\mathcal{O}(-\langle\theta,\check{\omega}_n\rangle x) \tag{29}$$

Following [4], for $\theta\in\Lambda_{G,P}$ let S^θ_P denote the ind-scheme classifying: $(\mathcal{F}_P,\nu)$, where $\mathcal{F}_P$ is a P-torsor on X and $\nu:\mathcal{F}_P\,\widetilde{\to}\,\mathcal{F}^0_P\mid_{X-x}$ is a trivialization such that $(\mathcal{F}_P\times_P Q,\nu)$ lies in $\mathrm{Gr}^\theta_{Q,x}$. In other words, S^θ_P classifies a P-torsor given by an exact sequence $0\to \mathrm{Sym}^2 L\to ?\to\Omega\to 0$ on X with $L\in\mathrm{Bun}_n$, a splitting of this sequence over $X-x$, and a trivialization $\nu: L\,\widetilde{\to}\,\mathcal{O}^n\mid_{X-x}$ with $\deg L=-\langle\theta,\check{\omega}_n\rangle$. The reduced part $(S^\theta_P)_{red}$ is given by the addidtional condition that ν induces an isomorphism (29).

We have a map $\mathfrak{s}^\theta_P: S^\theta_P\to \mathrm{Gr}_{G,x}$ sending $(\mathcal{F}_P,\nu)$ to $(\mathcal{F}_P\times_P G,\nu)$, its restriction $(S^\theta_P)_{red}\hookrightarrow \mathrm{Gr}_{G,x}$ is a locally closed immersion.

The map $\mathfrak{s}^\theta_{\bar P}: S^\theta_{\bar P}\to\mathrm{Gr}_{G,x}$ is defined in a similar way using the lagrangian subbundle $\Omega^n\subset\mathcal{O}^n_X\oplus\Omega^n$ that defines the opposite parabolic subgroup $\bar P\subset G$.

Write $\mathfrak{t}^\theta_P: S^\theta_P\to\mathrm{Gr}^\theta_{Q,x}$ for the projection sending $(\mathcal{F}_P,\nu)$ to $(\mathcal{F}_P\times_P Q,\nu)$ and $\mathfrak{r}^\theta_P:\mathrm{Gr}^\theta_{Q,x}\hookrightarrow S^\theta_P$ for the natural section, similarly for $\bar P$.

Fix an isomorphism $\mathbb{G}_m\,\widetilde{\to}\,Z(Q)$, where $Z(Q)$ is the center of Q, in such a way that $\mathbb{G}_m\,\widetilde{\to}\,Z(Q)$ acts adjointly on the unipotent radical $U(P)\subset P$ with strictly positive weights. The subscheme of $Z(Q)$-fixed points in Gr_G is $Q(K)G(\mathcal{O})/G(\mathcal{O})$, its connected components are $\mathrm{Gr}^\theta_{Q,red}$, $\theta\in\Lambda_{G,P}$. One checks that

$$\{x\in\mathrm{Gr}_{G,x}\mid \lim_{t\to 0} tx\in\mathrm{Gr}^\theta_{Q,x,red}\}=(S^\theta_P)_{red}\quad\text{and}$$

$$\{x\in\mathrm{Gr}_{G,x}\mid \lim_{t\to \infty} tx\in\mathrm{Gr}^\theta_{Q,x,red}\}=(S^\theta_{\bar P})_{red}$$

Consider the diagram

$$\begin{array}{ccc}
\widetilde{S}^\theta_P & \stackrel{\tilde{\mathfrak{s}}^\theta_P}{\to} & \widetilde{\mathrm{Gr}}_{G,x}\\
\uparrow \tilde{\mathfrak{r}}^\theta_P & & \uparrow \tilde{\mathfrak{s}}^\theta_{\bar P}\\
\widetilde{\mathrm{Gr}}^\theta_{Q,x} & \stackrel{\tilde{\mathfrak{r}}^\theta_{\bar P}}{\to} & \widetilde{S}^\theta_{\bar P}
\end{array}$$

obtained by restricting the gerbe $\widetilde{\mathrm{Gr}}_{G,x}\to\mathrm{Gr}_{G,x}$ with respect to the corresponding maps.

Lemma 11. *There exists a canonical $P(\mathcal{O}_x)$-equivariant section $i^\theta_P: S^\theta_P\to\widetilde{S}^\theta_P$ of the gerbe $\widetilde{S}^\theta_P\to S^\theta_P$.*

Proof Remind the line bundle $\mathcal{L}$ on $\mathrm{Gr}_{G,x}$ introduced in 8.3. Consider the map $\mathrm{Gr}_{G,x}\to\mathrm{Bun}_G$ sending $(\mathcal{F}_G,\nu:\mathcal{F}_G\,\widetilde{\to}\,\mathcal{F}^0_G\mid_{X-x})$ to $\mathcal{F}_G$. The restriction of $\mathcal{A}$ under this map identifies canonically with $\mathcal{L}^{-1}\otimes\det\mathrm{R}\Gamma(X,M_0)$, where $M_0=\mathcal{O}^n_X\oplus\Omega^n$. Since $\det\mathrm{R}\Gamma(X,M_0)\,\widetilde{\to}\,\det\mathrm{R}\Gamma(X,\mathcal{O})^{\otimes 2n}$,

we get a cartesian square

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}_{G,x} & \to & \widetilde{\mathrm{Bun}}_G \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{G,x} & \to & \mathrm{Bun}_G \end{array}$$

Remind the map $\tilde{\nu}$ defined in Lemma 5. Now the diagram

$$\begin{array}{ccccc} S^\theta_P & \to & \mathrm{Bun}_P & \stackrel{\tilde{\nu}}{\to} & \widetilde{\mathrm{Bun}}_G \\ \downarrow & & \downarrow & \swarrow \mathfrak{r} & \\ \mathrm{Gr}_{G,x} & \to & \mathrm{Bun}_G & & \end{array}$$

yields the section i^θ_P.

To see that it is $P(\mathcal{O}_x)$-equivariant, rewrite it in local terms as follows. On $\mathrm{Gr}^\theta_{Q,x}$ we have the $\mathbb{Z}/2\mathbb{Z}$-graded $Q(\mathcal{O}_x)$-equivariant line bundle, say ${}_\theta\mathcal{L}$, whose fibre at $(L, L \widetilde{\to} \mathcal{O}^n \mid_{X-x})$ is

$$\det(L_0 \otimes \mathcal{O}_x : L \otimes \mathcal{O}_x)$$

with $L_0 = \mathcal{O}^n_X$. Hence $(\mathfrak{t}^\theta_P)^* {}_\theta\mathcal{L}$ is a $P(\mathcal{O}_x)$-equivariant line bundle on S^θ_P. The canonical $\mathbb{Z}/2\mathbb{Z}$-graded $P(\mathcal{O}_x)$-equivariant isomorphism $(\mathfrak{s}^\theta_P)^*\mathcal{L} \widetilde{\to} (\mathfrak{t}^\theta_P)^*({}_\theta\mathcal{L})^{\otimes 2}$ defines the section i^θ_P via 3.1.2. $\square$

Define the functors $F^\theta, F'^\theta : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x}) \to \mathrm{D}(\mathrm{Gr}^\theta_{Q,x})$ by

$$F'^\theta(K) = (\mathfrak{t}^\theta_P)_!(i^\theta_P)^*(\tilde{\mathfrak{s}}^\theta_P)^*K \quad \text{and} \quad F^\theta(K) = F'^\theta(K) \otimes \bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\otimes\langle\theta, 2\check{\rho}-2\check{\rho}_Q\rangle}$$

We have used the fact that $2(\check{\rho} - \check{\rho}_Q) \in \check{\Lambda}_{G,P}$.

Remark 7. We could replace in the definition of F^θ and F'^θ the ind-schemes S^θ_P and $\mathrm{Gr}^\theta_{Q,x}$ by their reduced parts, the corresponding functors would be canonically isomorphic to the old ones. In some geometric questions we work rather with the corresponding reduced ind-schemes (without indicating that explicitly, for example in Proposition 12 and 15, Corolary 1 and so on).

Proposition 12. *The functor $F^\theta(K)$ maps $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ to the category $\mathrm{Sph}(\mathrm{Gr}^\theta_{Q,x})$ of $Q(\mathcal{O}_x)$-equivariant perverse sheaves on $\mathrm{Gr}^\theta_{Q,x}$. In particular, it is exact.*

Proof By Lemma 9 combined with Proposition 19, we get the hyperbolic localization functors $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x}) \to \mathrm{D}(\widetilde{\mathrm{Gr}}^\theta_{Q,x})$ given by

$$K \mapsto (\tilde{\mathfrak{t}}^\theta_{\bar{P}})^*(\tilde{\mathfrak{s}}^\theta_{\bar{P}})^!K \widetilde{\to} (\tilde{\mathfrak{t}}^\theta_P)^!(\tilde{\mathfrak{s}}^\theta_P)^*K = K^{!*} \tag{30}$$

By Lemma 11, we have moreover $K^{!*} \widetilde{\to} (\mathfrak{t}^\theta_P \times \mathrm{id})_!(\tilde{s}^\theta_P)^*K$, where

$$\mathfrak{t}^\theta_P \times \mathrm{id} : \tilde{S}^\theta_P = S^\theta_P \times B(\mu_2) \to \mathrm{Gr}^\theta_{Q,x} \times B(\mu_2) = \widetilde{\mathrm{Gr}}^\theta_{Q,x}$$

The complex $K^{!*}$ is $Q(\mathcal{O}_x)$-equivariant, because both $\tilde{\mathfrak{s}}^\theta_P$ and $\tilde{\mathfrak{t}}^\theta_P$ are $Q(\mathcal{O}_x)$-equivariant. The dimension estimates given in ([4], Proposition 4.3.3) show that $F^\theta(K)$ is placed in non-positive perverse degrees. Now (30) garantees that $F^\theta(K)$ is placed in non-negative perverse degrees. $\square$

Let w_0 (resp., w^Q_0) denote the longest element of the Weil group W of G (resp., W_Q of Q).

Corollary 1. *i) Let $\lambda \in \Lambda^+$ and θ be the image of λ in $\Lambda_{G,P}$. Then $\mathcal{A}_{Q,\lambda}$ (resp., $\mathcal{A}_{Q,-w_0^Q(\lambda)}$) appears with multiplicity one in $F^\theta(\mathcal{A}_\lambda)$ (resp., in $F^{-\theta}(\mathcal{A}_\lambda)$).*
ii) The functor $F : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x}) \to \mathrm{Sph}(\mathrm{Gr}_{Q,x})$ given by $F = \underset{\theta\in\Lambda_{G,P}}{\oplus} F^\theta$ is exact and faithful.

Proof i) Note that $S_P^\theta \cap \mathrm{Gr}_G^\lambda$ is open in Gr_G^λ. Moreover, $\mathrm{Gr}_Q^\theta \cap \mathrm{Gr}_G^\lambda = \mathrm{Gr}_Q^\lambda$. Since P/Q is affine, $\mathrm{Gr}_Q \hookrightarrow S_P$ is a closed immersion. So, $\mathrm{Gr}_Q^\theta \cap \mathrm{Gr}_G^\lambda \hookrightarrow S_P^\theta \cap \mathrm{Gr}_G^\lambda$ is a smooth closed subscheme. It follows that $(\tilde{\mathfrak{r}}_P^\theta)^!(\tilde{\mathfrak{s}}_P^\theta)^*\mathcal{A}_\lambda$ is a shifted constant sheaf over Gr_Q^λ. The first assertion follows.

For the second, note that $\mathrm{Gr}_Q^{-\theta} \cap \mathrm{Gr}_G^\lambda = \mathrm{Gr}_Q^{-w_0^Q(\lambda)}$, and the map

$$\mathfrak{t}_P^{-\theta} : S_P^{-\theta} \cap \mathrm{Gr}_G^\lambda \to \mathrm{Gr}_Q^{-\theta}$$

is an isomorphism over the $Q(\mathcal{O})$-orbit $\mathrm{Gr}_Q^{-w_0^Q(\lambda)}$.
ii) Since F is exact, to show faithfulness, it suffices to prove that F does not annihilate a nonzero object. To this end, it suffices to show that $F(\mathcal{A}_\lambda) \neq 0$ for any dominant coweight λ, which follows from i). $\square$

8.5 Example: explicit calculation Let $\alpha \in \Lambda^+$ denote the coroot of $\mathbb{S}\mathrm{p}_{2n}$ corresponding to the maximal root $\check{\alpha}_{max}$ of $\mathbb{S}\mathrm{p}_{2n}$. So, α is the highest weight of the standard representation of the Langlands dual group $\mathrm{S}\mathbb{O}_{2n+1}$ of $\mathbb{S}\mathrm{p}_{2n}$. For this subsection take G to be that of 8.1 for $M_0 = \mathcal{O}^n \oplus \Omega_{\mathcal{O}}^n$. Following ([6], Sect. 4.5.12) the closure $\overline{\mathrm{Gr}}_G^\alpha$ of Gr_G^α in Gr_G is described as follows.

The $G(k)$-orbit V in Gr_G passing through $\alpha(t)G(\mathcal{O})$ is identified with the projective space $V \widetilde{\to} \mathbb{P}^{2n-1}$, and Gr_G^α is the total space of the line bundle $\mathcal{O}(2)$ over V.

Let $V = \mathbb{P}^{2n-1} \hookrightarrow \mathbb{P}^{n(2n+1)-1}$ be the Veronese map. Write $x_1, \ldots, x_{2n}$ for the homogeneous coordinates in $\mathbb{P}^{2n-1}$ and t_{ij} with $1 \le i \le j \le 2n$ for the homogeneous coordinates in $\mathbb{P}^{n(2n+1)-1}$. Then the inclusion is given by $t_{ij} = x_i x_j$. Its image is the subscheme defined by homogeneous equations

$$t_{ij}t_{kl} = t_{ik}t_{jl} \tag{31}$$

for all i, j, k, l whenever this makes sense.

One may identify the Lie algebra of $\mathbb{S}\mathrm{p}_{2n}$ with $\mathbb{A}^{n(2n+1)}$ in such a way that the set Z of elements $\mathbb{S}\mathrm{p}_{2n}$-conjugate to a multiple of the maximal root becomes the subscheme $Z \subset \mathbb{A}^{n(2n+1)} = \operatorname{Spec} k[t_{ij}]$ given by equations (31). Let $A \in Z$ denote the origin of this cone. Let $\bar{Z} \subset \mathbb{P}^{n(2n+1)}$ be the projective closure of Z. Then $\overline{\mathrm{Gr}}_G^\alpha = \bar{Z}$ and $\mathrm{Gr}_G^\alpha = \bar{Z} - A$.

The projection $\pi : \bar{Z} - A \to V$ is an affine fibration on which $\mathcal{O}(2)$ acts transitively and freely (and the corresponding torsor is trivial). So, π^* yields a diagram of isomorphisms

$$\begin{array}{ccccc}
\mathrm{Cl}(V) & \widetilde{\to} & \mathrm{Cl}(\bar{Z} - A) & \widetilde{\to} & \mathrm{Cl}(\bar{Z}) \\
\downarrow & & \downarrow & & \\
\mathrm{Pic}(V) & \widetilde{\to} & \mathrm{Pic}(\bar{Z} - A) & \widetilde{\to} & \mathbb{Z},
\end{array}$$

where for a variety S we denote by $\mathrm{Cl}(S)$ the Weil divisors class group.

Write (t_{ij}, w) for the homogeneous coordinates in $\mathbb{P}^{n(2n+1)}$. Let the subscheme $V \subset \bar{Z}$ be given by $w = 0$, it is a section of π. We have $Z = \bar{Z} - V$.

The image in $\mathrm{Cl}(V)$ of the hyperplane section of $\mathbb{P}^{n(2n+1)-1}$ is 2. It follows that the image of V in $\mathrm{Cl}(\bar{Z})$ is 2 and $\mathrm{Cl}(Z) \widetilde{\to} \mathbb{Z}/2\mathbb{Z}$.

Let $L \subset Z$ denote the preimage under π of the subscheme of V given by $x_1 = 0$. Denote again by L the corresponding Weil divisor on $\bar{Z}$. Then L is not locally principal in $\mathcal{O}_{Z,A}$. Indeed, let $\mathfrak{p} \subset \mathcal{O}_{Z,A}$ denote the ideal corresponding to L and $\mathfrak{m}_{Z,A} \subset \mathcal{O}_{Z,A}$ the maximal ideal. Then t_{ij} $(1 \le i \le j \le n)$ form a base in the cotangent space $\mathfrak{m}_{Z,A}/\mathfrak{m}^2_{Z,A}$, and the elements $t_{1j} \in \mathfrak{p}$ $(1 \le j \le n)$ are linearly independent in $\mathfrak{m}_{Z,A}/\mathfrak{m}^2_{Z,A}$. So, $\mathrm{Pic}\, Z = 0$, and $\mathcal{O}_{\bar{Z}}(V)$ generates $\mathrm{Pic}(\bar{Z})$. The image of $\mathcal{O}_{\bar{Z}}(V)$ under the composition

$$\mathrm{Pic}(\bar{Z}) \hookrightarrow \mathrm{Cl}(\bar{Z}) \widetilde{\to} \mathrm{Cl}(\bar{Z} - A) \widetilde{\to} \mathrm{Pic}(\bar{Z} - A) \widetilde{\to} \mathbb{Z}$$

is 2. In other words, $\mathcal{O}_{\bar{Z}-A}(L)$ does not extend to a line bundle on $\bar{Z}$.

The line bundle $\mathcal{L} \mid_{\overline{\mathrm{Gr}}^{\alpha}_G}$ identifies with $\mathcal{O}_{\mathbb{P}^{n(2n+1)}}(1) \mid_{\bar{Z}}$. Let $\tilde{Z} \to \bar{Z}$ denote the μ_2-gerbe of square roots of this bundle. We see that this gerbe is nontrivial, though trivial over $\bar{Z} - A$.

Set $Y = \mathbb{A}^{2n} = \mathrm{Spec}\, k[x_i]$. Let $\tau : Y \to Z$ be the map given by $t_{ij} = x_i x_j$. Clearly, $Y - \tau^{-1}(A) \to Z - A$ is a S_2-Galois covering.

For a coweight λ of Q denote by $\mathcal{A}_{Q,\lambda}$ the intersection cohomology sheaf of the $Q(\mathcal{O})$-orbit on Gr_Q passing through $\lambda(t)Q(\mathcal{O})$.

Proposition 13. *1) The sheaf $\mathcal{A}_\alpha$ is the extension by zero from $\bar{Z} - A$.*
2) We have $F^0(\mathcal{A}_\alpha) = 0$. For $\theta \in \Lambda_{G,P}$ such that $\langle \theta, \check{\omega}_n \rangle = 1$ we have $F^\theta(\mathcal{A}_\alpha) \widetilde{\to} \mathcal{A}_{Q,\alpha}$ and $F^{-\theta}(\mathcal{A}_\alpha) \widetilde{\to} \mathcal{A}_{Q,-\alpha}$.

Proof 1) Note that $\mathcal{O}_{Z-A}(L)$ generates the group $\mathrm{Pic}(Z - A) \widetilde{\to} \mathrm{Cl}(Z - A) \widetilde{\to} \mathrm{Cl}(Z) \widetilde{\to} \mathbb{Z}/2\mathbb{Z}$. The gerbe $\tilde{Z}$ is obtained by gluing together trivial gerbes $Z \times B(\mu_2)$ and $(\bar{Z} - A) \times B(\mu_2)$ over $Z - A$. The gluing data is an automorphism of the gerbe $(Z - A) \times B(\mu_2)$ which can be described as follows.

An S-point of $(Z - A) \times B(\mu_2)$ is a line bundle $\mathcal{B}$ on S together with $\mathcal{B}^2 \widetilde{\to} \mathcal{O}_S$ and a map $S \to (Z - A)$. Our automorphism sends this point to the same map $S \to (Z - A)$ and replaces $\mathcal{B}$ by $\mathcal{B}$ tensored with the restriction of $\mathcal{O}_{Z-A}(L)$ to S.

We have the μ_2-torsor over $Z - A$ consisting of those sections of $\mathcal{O}_{Z-A}(L)$ whose square is 1. This is exactly the Galois covering $Y - \tau^{-1}(A) \to Z - A$.

Let W denote the nontrivial rank one local system on $B(\mu_2)$ corresponding to the covering $\mathrm{Spec}\, k \to B(\mu_2)$. If we identify our gerbe over Z with $Z \times B(\mu_2)$ then over that locus $\mathcal{A}_\alpha$ becomes the exteriour product $N \boxtimes W$, where N is the nontrivial local system on $Z - A$ extended by zero to A and corresponding to the covering $Y - \tau^{-1}(A) \to Z - A$.

2) Considering Gr^0_Q as a subscheme of Gr_G, one checks that $\mathrm{Gr}^0_Q \cap \overline{\mathrm{Gr}}^{\alpha}_G$ is the point scheme $1 \in \mathrm{Gr}_G$. Consider the $*$-restriction $N \mid_{Z \cap L}$. Since the !-fibre at A of $N \mid_{Z \cap L}$ vanishes, we get $F^0(\mathcal{A}_\alpha) = 0$.

Let $\theta \in \Lambda_{G,P}$ be such that $\langle \theta, \check{\omega}_n \rangle = 1$. Recall the map $\pi : \bar{Z} - A \to V$. We have

$$\mathrm{Gr}^{\alpha}_G \cap S^{\theta}_P = \pi^{-1}(V_0),$$

where $V_0 \subset V = \mathbb{P}(M_0(x)/M_0)$ is the complement to $\mathbb{P}(L_0(x)/L_0)$. In other words, $\mathrm{Gr}_G^\alpha \cap S_P^\theta \subset \mathrm{Gr}_G^\alpha$ is the open subscheme given by the condition that the line $(M+M_0)/M_0$ is not contained in $L_0(x)/L_0$. Further, $\mathrm{Gr}_G^\alpha \cap \mathrm{Gr}_Q^\theta = \mathrm{Gr}_Q^\alpha$. The isomorphism $F^\theta(\mathcal{A}_\alpha) \widetilde{\to} \mathcal{A}_{Q,\alpha}$ follows.

We have $\mathrm{Gr}_G^\alpha \cap S_P^{-\theta} = \mathrm{Gr}_Q^{-\alpha}$. This yields the last isomorphism. $\square$

Remark 8. Let $\lambda \in \Lambda^+$ and $\theta \in \Lambda_{G,P}$. If $F^\theta(\mathcal{A}_\lambda) \neq 0$ then

$$-\langle \lambda, \check{\omega}_n \rangle \leq \langle \theta, \check{\omega}_n \rangle \leq \langle \lambda, \check{\omega}_n \rangle \tag{32}$$

Indeed, if $S_P^\theta \cap \overline{\mathrm{Gr}}_G^\lambda \neq \emptyset$ then (32) holds. More generally, for a reductive group G and its parabolic subgroup P the condition $S_P^\theta \cap \overline{\mathrm{Gr}}_G^\lambda \neq \emptyset$ implies $\langle \lambda, w_0(\check{\lambda}) \rangle \leq \langle \theta, \check{\lambda} \rangle \leq \langle \lambda, \check{\lambda} \rangle$ for any $\check{\lambda} \in \check{\Lambda}_{G,P}$ which is dominant for G.

8.6 The functors $F^\theta_{X^d}$

Let Gr_{Q,X^d} denote the ind-scheme classifying $(x_1, \ldots, x_d) \in X^d$ and $L \in \mathrm{Bun}_n$ with trivialization $L \widetilde{\to} \mathcal{O}^n \mid_{X - x_1 \cup \ldots \cup x_d}$. Its connected components are indexed by $\Lambda_{G,P}$, the component $\mathrm{Gr}^\theta_{Q,X^d}$ is given by $\deg L = -\langle \theta, \check{\omega}_n \rangle$. We have a natural map $\mathrm{Gr}_{Q,X^d} \to \mathrm{Gr}_{G,X^d}$ sending the above point to $L \oplus (L^* \otimes \Omega)$ with the induced trivialization outside x_i. The composition

$$(\mathrm{Gr}_{Q,X^d})_{red} \hookrightarrow \mathrm{Gr}_{Q,X^d} \to \mathrm{Gr}_{G,X^d}$$

is a closed immersion.

For $\theta \in \Lambda_{G,P}$ denote by S^θ_{P,X^d} the ind-scheme classifying collections: $(x_1, \ldots, x_d) \in X^d$, a P-torsor $\mathcal{F}_P$ on X with trivialization $\nu : \mathcal{F}_P \widetilde{\to} \mathcal{F}_P^0 \mid_{X - x_1 \cup \ldots \cup x_d}$ such that the induced Q-torsor $\mathcal{F}_P \times_P Q$ lies in $\mathrm{Gr}^\theta_{Q,X^d}$. Here $\mathcal{F}^0_P$ is the G-torsor $\mathcal{F}^0_G = \mathcal{O}_X^n \oplus \Omega^n$ with P-structure corresponding to the lagrangian subbundle $\mathcal{O}_X^n$.

Considering $\mathcal{F}^0_{\bar{P}}$ as $\mathcal{F}^0_G$ with $\bar{P}$-structure given by Ω^n, one similarly defines the ind-scheme $S^\theta_{\bar{P},X^d}$. As in 8.4, one defines a diagram

$$\begin{array}{ccc} S^\theta_{P,X^d} & \overset{\mathfrak{s}^\theta_{P,X^d}}{\to} & \mathrm{Gr}_{G,X^d} \\ \uparrow \mathfrak{t}^\theta_{P,X^d} & & \uparrow \\ \mathrm{Gr}^\theta_{Q,X^d} & \to & S^\theta_{\bar{P},X^d} \end{array} \tag{33}$$

Both $(S^\theta_{P,X^d})_{red}$ and $(S^\theta_{\bar{P},X^d})_{red}$ are locally closed in Gr_{G,X^d}, and their intersection is $(\mathrm{Gr}^\theta_{Q,X^d})_{red}$.

For a k-point $(x_1, \ldots, x_d) \in X^d$ with $\{x_1, \ldots, x_d\} = \{y_1, \ldots, y_s\}$ and y_i pairwise distinct, the fibre of the diagram (33) over $(x_1, \ldots, x_d) \in X^d$ is

$$\begin{array}{ccc} \underset{\theta_1 + \ldots + \theta_s = \theta}{\cup} (\prod_i S_P^{\theta_i}) & \to & \prod_{i=1}^s \mathrm{Gr}_{G,y_i} \\ \uparrow & & \uparrow \\ \underset{\theta_1 + \ldots + \theta_s = \theta}{\cup} (\prod_i \mathrm{Gr}^{\theta_i}_{Q,y_i}) & \to & \underset{\theta_1 + \ldots + \theta_s = \theta}{\cup} (\prod_i S_{\bar{P}}^{\theta_i}) \end{array}$$

Similarly to G_{X^d}, one defines a group scheme Q_{X^d} (resp., P_{X^d}) over X^d, it acts naturally on $\mathrm{Gr}^{\theta}_{Q,X^d}$ (resp., on S^{θ}_{P,X^d}). Denote by $\mathrm{Sph}(\mathrm{Gr}^{\theta}_{Q,X^d})$ the category of Q_{X^d}-equivariant perverse sheaves on $\mathrm{Gr}^{\theta}_{Q,X^d}$. Let us define the functors

$$F^{\theta}_{X^d}, F'^{\theta}_{X^d} : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,X^d}) \to \mathrm{D}(\mathrm{Gr}^{\theta}_{Q,X^d})$$

Let $\tilde{\mathfrak{s}}^{\theta}_{P,X^d} : \tilde{S}^{\theta}_{P,X^d} \to \widetilde{\mathrm{Gr}}_{G,X^d}$ be the map obtained by the base change $\widetilde{\mathrm{Gr}}_{G,X^d} \to \mathrm{Gr}_{G,X^d}$ from (33). As in Lemma 11, one defines a P_{X^d}-equivariant section $i^{\theta}_{P,X^d} : S^{\theta}_{P,X^d} \to \tilde{S}^{\theta}_{P,X^d}$ of the gerbe $\tilde{S}^{\theta}_{P,X^d} \to S^{\theta}_{P,X^d}$. We have a Q_{X^d}-equivariant line bundle ${}_{\theta}\mathcal{L}_{X^d}$ on $\mathrm{Gr}^{\theta}_{Q,X^d}$, whose fibre at

$$(L, L \,\widetilde{\to}\, \mathcal{O}^n \mid_{X - x_1 \cup \ldots \cup x_d})$$

is $\det \mathrm{R}\Gamma(X, \mathcal{O}^n_X) \otimes \det \mathrm{R}\Gamma(X, L)^{-1}$. As $\mathbb{Z}/2\mathbb{Z}$-graded, it is placed in degree $\flat(\theta) := \langle \theta, \check{\omega}_n \rangle \mod 2$. The canonical P_{X^d}-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$(\mathfrak{s}^{\theta}_{P,X^d})^* \mathcal{L} \,\widetilde{\to}\, {}_{\theta}\mathcal{L}^{\otimes 2}_{X^d} \mid_{S^{\theta}_{P,X^d}}$$

yields i^{θ}_{P,X^d} via 3.1.2. Set

$$F'^{\theta}_{X^d}(K) = (\mathfrak{r}^{\theta}_{P,X^d})^! (i^{\theta}_{P,X^d})^* (\tilde{\mathfrak{s}}^{\theta}_{P,X^d})^* K \quad \text{and} \quad F^{\theta}_{X^d}(K) = F'^{\theta}_{X^d}(K) \otimes \bar{\mathbb{Q}}_{\ell}[1](\frac{1}{2})^{\otimes \langle \theta, 2\check{\rho} - 2\check{\rho}_Q \rangle}$$

Note that

$$F'^{\theta}_{X^d}(K) \,\widetilde{\to}\, (\mathfrak{t}^{\theta}_{P,X^d})_! (i^{\theta}_{P,X^d})^* (\tilde{\mathfrak{s}}^{\theta}_{P,X^d})^* K$$

where $\mathfrak{t}^{\theta}_{P,X^d} : S^{\theta}_{P,X^d} \to \mathrm{Gr}^{\theta}_{Q,X^d}$ is the corresponding contraction map.

Remind the definition of the tensor category $\mathrm{Sph}(\mathrm{Gr}_{Q,x})^{\natural}$. Equip $\mathrm{Sph}(\mathrm{Gr}_{Q,x})$ with the convolution product, associativity and commutativity constraints given by the fusion procedure, then $\mathrm{Sph}(\mathrm{Gr}_{Q,x})$ is a tensor category ([6], 5.3.16). It has a canonical $\mathbb{Z}/2\mathbb{Z}$-grading compatible with the tensor structure, namely $\mathcal{A}_{Q,\lambda}$ is even (resp., odd) if $\dim \mathrm{Gr}^{\lambda}_Q$ is even (resp., odd). The latter condition depends only on the connected component of $\mathrm{Gr}_{Q,x}$ containing $\mathrm{Gr}^{\lambda}_{Q,x}$.

Following ([6], 5.3.21), we define $\mathrm{Sph}(\mathrm{Gr}_{Q,x})^{\natural}$ as the full subcategory of even objects in $\mathrm{Sph}(\mathrm{Gr}_{Q,x}) \otimes \mathrm{Vect}^{\epsilon}$. We have an equivalence of monoidal categories $\mathrm{Sph}(\mathrm{Gr}_{Q,x})^{\natural} \to \mathrm{Sph}(\mathrm{Gr}_{Q,x})$ (i.e., it is compatible with tensor product and associativity constraints), and the commutativity constraints $A \otimes B \,\widetilde{\to}\, B \otimes A$ in these two categories differ by $(-1)^{\deg A \deg B}$.

Let $h^{\epsilon} : \mathrm{Sph}(\mathrm{Gr}_{Q,x}) \to \mathrm{Vect}^{\epsilon}$ denote the global cohomology functor. Since h^{ϵ} is a tensor functor compatible with $\mathbb{Z}/2\mathbb{Z}$-gradings, it gives rise to a tensor functor

$$h : \mathrm{Sph}(\mathrm{Gr}_{Q,x})^{\natural} \to \mathrm{Vect}$$

By [17], h is a fibre functor, and there is an isomorphism $\mathrm{Aut}^{\otimes} h \,\widetilde{\to}\, \check{Q}$, where $\check{Q}$ is the Langlands dual group to Q (in [6], 5.3.23 some properties of the action of $\check{Q}$ on h are listed, which determine this isomorphism uniquely). Thus, $\mathrm{Sph}(\mathrm{Gr}_{Q,x})^{\natural} \,\widetilde{\to}\, \mathrm{Rep}(\check{Q})$ canonically as tensor categories.

Consider

$$\mathrm{Sph}'(\mathrm{Gr}_{Q,x}) := \underset{\theta\in\Lambda_{G,P}}{\oplus} \mathrm{Sph}(\mathrm{Gr}^{\theta}_{Q,x})[\langle\theta, 2\check{\rho}_Q - 2\check{\rho}\rangle] \subset \mathrm{D}(\mathrm{Gr}_{Q,x}) \tag{34}$$

equipped with the convolution product, commutativity and associativity constraints given by the fusion procedure, so $\mathrm{Sph}'(\mathrm{Gr}_{Q,x})$ is a tensor category.

Lemma 12. *There is a canonical equivalence of tensor categories* $\mathrm{Sph}'(\mathrm{Gr}_Q) \widetilde{\to} \mathrm{Sph}(\mathrm{Gr}_Q)^{\natural}$.

Proof Note that $2(\check{\rho} - \check{\rho}_Q) = (n+1)\check{\omega}_n \in \check{\Lambda}_{G,P}$. Consider the case of n odd. In this case $\check{\rho}_Q \in \check{\Lambda}$, so all $Q(\mathcal{O})$-orbits on Gr_Q are even-dimensional and $\mathrm{Sph}(\mathrm{Gr}_G) \widetilde{\to} \mathrm{Sph}(\mathrm{Gr}_G)^{\natural}$. In this case the shifts in (34) are even, and we are done.

Consider the case of n even. The component $\mathrm{Gr}^{\theta}_{Q,x}$ is even iff $\langle\theta, \check{\omega}_n\rangle$ is even. So, in (34) the even (resp, odd) objects of $\mathrm{Sph}(\mathrm{Gr}_{Q,x})$ are shifted by even (resp., odd) cohomological degree. Our assertion follows. $\square$

Equip $\mathrm{Sph}'(\mathrm{Gr}_{Q,x})$ with a new $\mathbb{Z}/2\mathbb{Z}$-grading such that $K \in \mathrm{Sph}'(\mathrm{Gr}^{\theta}_{Q,x})$ is placed in degree $\flat(\theta)$. This $\mathbb{Z}/2\mathbb{Z}$-grading is compatible with the tensor structure. Denote by $\mathrm{Sph}'(\mathrm{Gr}_{Q,x})^{\flat}$ the category of even objects in $\mathrm{Sph}'(\mathrm{Gr}_{Q,x}) \otimes \mathrm{Vect}^{\epsilon}$, it is equipped with the induced $\mathbb{Z}/2\mathbb{Z}$-grading.

The proof of part ii) of the following proposition is postponed to Sect. 8.7.

Proposition 14. *i) The functor* $F' : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x}) \to \mathrm{Sph}'(\mathrm{Gr}_{Q,x})^{\flat}$ *given by* $F' = \underset{\theta\in\Lambda_{G,P}}{\oplus} F'^{\theta}$ *is a tensor functor.*
ii) There is a unique $\mathbb{Z}/2\mathbb{Z}$*-grading on* $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})$ *such that* F' *is compatible with* $\mathbb{Z}/2\mathbb{Z}$*-gradings.*

Proof i) Pick $F_1, F_2 \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. Set $K_i = \tau^0 F_i$,

$$K = F^{\theta}_{X^2}(K_1 *_X K_2) \quad \text{and} \quad K' = F'^{\theta}_{X^2}(K_1 *_X K_2),$$

where τ^0 is given by (26). By abuse of notation, write also $\tau^0 : \mathrm{Sph}(\mathrm{Gr}_Q) \to \mathrm{Sph}(\mathrm{Gr}_{Q,X})[-1]$ for the corresponding functor for Q.

Step 1. Recall that $U \subset X^2$ denotes the complement to the diagonal. Write $\widetilde{\mathrm{Gr}}_{G,X^2}(U)$ for the preimage of U in $\widetilde{\mathrm{Gr}}_{G,X^2}$. We have a μ_2-gerbe $q : (\widetilde{\mathrm{Gr}}_{G,X} \times \widetilde{\mathrm{Gr}}_{G,X}) \mid_U \to \widetilde{\mathrm{Gr}}_{G,X^2}(U)$ (defined as the map $\tilde{q}_{G,X}$ in 8.3.1). The complex $q^*(K_1 *_X K_2)$ identifies canonically with $(K_1 \boxtimes K_2) \mid_U$. Denote by i^{θ} the composition

$$S^{\theta}_{P,X^d} \overset{i^{\theta}_{P,X^d}}{\to} \tilde{S}^{\theta}_{P,X^d} \overset{\tilde{\mathfrak{s}}^{\theta}_{P,X^d}}{\to} \widetilde{\mathrm{Gr}}_{G,X^d}$$

For $\theta_1 + \theta_2 = \theta$ the following diagram is 2-commutative

$$\begin{array}{ccc} (\widetilde{\mathrm{Gr}}_{G,X} \times \widetilde{\mathrm{Gr}}_{G,X}) \mid_U & \overset{q}{\to} & \widetilde{\mathrm{Gr}}_{G,X^2}(U) \\ \uparrow i^{\theta_1} \times i^{\theta_2} & & \uparrow i^{\theta} \\ (S^{\theta_1}_{P,X} \times S^{\theta_2}_{P,X}) \mid_U & \hookrightarrow & S^{\theta}_{P,X^2}(U), \end{array}$$

where the low horizontal arrow is the natural open immersion. However, the 2-morphism rending this diagram 2-commutative is well-defined only up to a sign, we normalize it as follows.

Write ${}_\theta\underline{\mathcal{L}}_{X^d}$ for the line bundle ${}_\theta\mathcal{L}_{X^d}$ viewed as *ungraded*. It suffices to pick an isomorphism

$$\epsilon^{\theta_1,\theta_2} : {}_{\theta_1}\underline{\mathcal{L}}_X \boxtimes {}_{\theta_2}\underline{\mathcal{L}}_X \widetilde{\to} (j^{\theta_1,\theta_2})^* {}_\theta\underline{\mathcal{L}}_{X^2},$$

where $j^{\theta_1,\theta_2} : (\mathrm{Gr}^{\theta_1}_{Q,X} \times \mathrm{Gr}^{\theta_2}_{Q,X}) \mid_U \hookrightarrow \mathrm{Gr}^\theta_{Q,X^2}(U)$ is the natural open immersion. The order of points in X^2 yields such $\epsilon^{\theta_1,\theta_2}$, and the usual Leibnitz rule is satisfied.

Namely, remind that σ denotes the involution of X^2 permuting the points. For the diagram

$$\begin{array}{ccc}
(\mathrm{Gr}^{\theta_1}_{Q,X} \times \mathrm{Gr}^{\theta_2}_{Q,X}) \mid_U & \stackrel{j^{\theta_1,\theta_2}}{\to} & \mathrm{Gr}^\theta_{Q,X^2}(U) \\
\uparrow \sigma & & \uparrow \sigma \\
(\mathrm{Gr}^{\theta_2}_{Q,X} \times \mathrm{Gr}^{\theta_1}_{Q,X}) \mid_U & \stackrel{j^{\theta_2,\theta_1}}{\to} & \mathrm{Gr}^\theta_{Q,X^2}(U)
\end{array}$$

the following diagram commutes

$$\begin{array}{ccccc}
\sigma^*(j^{\theta_1,\theta_2})^* {}_\theta\underline{\mathcal{L}}_{X^2} & \widetilde{\to} & (j^{\theta_2,\theta_1})^*\sigma^* {}_\theta\underline{\mathcal{L}}_{X^2} & \widetilde{\to} & (j^{\theta_2,\theta_1})^* {}_\theta\underline{\mathcal{L}}_{X^2} \\
\uparrow \epsilon & & & & \downarrow \mathrm{sign} \\
\sigma^*({}_{\theta_1}\underline{\mathcal{L}}_X \boxtimes {}_{\theta_2}\underline{\mathcal{L}}_X) & \widetilde{\to} & {}_{\theta_2}\underline{\mathcal{L}}_X \boxtimes {}_{\theta_1}\underline{\mathcal{L}}_X & \stackrel{\epsilon}{\to} & (j^{\theta_2,\theta_1})^* {}_\theta\underline{\mathcal{L}}_{X^2},
\end{array} \tag{35}$$

where sign $= (-1)^{\flat(\theta_1)\flat(\theta_2)}$, and the isomorphisms denoted by $\widetilde{\to}$ are the canonical ones.

Step 2. Note that $\mathrm{Gr}^\theta_{Q,X^2}(U)$ is the disjoint union of $(\mathrm{Gr}^{\theta_1}_{Q,X} \times \mathrm{Gr}^{\theta_2}_{Q,X}) \mid_U$ for $\theta_1 + \theta_2 = \theta$. Let us show that $K[2]$ is a perverse sheaf on $\mathrm{Gr}^\theta_{Q,X^2}$, the Goresky-MacPherson extension from $\mathrm{Gr}^\theta_{Q,X^2}(U)$. More precisely, we show that ϵ as above yield an isomorphism

$$(\tau^0 F'(F_1)) *_X (\tau^0 F'(F_2)) \widetilde{\to} F'_{X^2}(K_1 *_X K_2) \tag{36}$$

Indeed, $\epsilon^{\theta_1,\theta_2}$ yields an isomorphism between the restriction of K' to $(\mathrm{Gr}^{\theta_1}_{Q,X} \times \mathrm{Gr}^{\theta_2}_{Q,X}) \mid_U$ and

$$\tau^0 F'^{\theta_1}(F_1) \boxtimes \tau^0 F'^{\theta_2}(F_2)$$

So, $K[2]$ is a perverse sheaf over $\mathrm{Gr}^\theta_{Q,X^2}(U)$. Using (28), we learn that the $*$-restriction of K under the diagonal embedding $\mathrm{Gr}_{Q,X} \hookrightarrow \mathrm{Gr}_{Q,X^2}$ identifies with $\tau^0 F^\theta(F_1 * F_2)$, so it is placed in perverse degree 1. Now argue as in Proposition 12, using the corresponding $\mathbb{G}_m$-action on $\widetilde{\mathrm{Gr}}_{G,X^2}$. By Proposition 19, the !-restriction of K under $\mathrm{Gr}_{Q,X} \hookrightarrow \mathrm{Gr}_{Q,X^2}$ is placed in perverse degree 3. We have constructed the isomorphism (36).

Restricting to the diagonal, it yields $\tau^0(F'(F_1) * F'(F_2)) \widetilde{\to} \tau^0 F'(F_1 * F_2)$.

Step 3. Let us check the compatibility with the commutativity constraints. Using (35) one shows that the diagram commutes

$$\begin{array}{ccc}
\sigma^*(\tau^0 F'(F_1) *_X \tau^0 F'(F_2)) & \stackrel{\sigma^*\circ\epsilon}{\to} & \sigma^* F'_{X^2}(K_1 *_X K_2) \\
\uparrow & & \uparrow \\
\tau^0 F'(F_2) *_X \tau^0 F'(F_1) & \stackrel{\mathrm{sign}\circ\epsilon}{\to} & F'_{X^2}(K_2 *_X K_1),
\end{array}$$

where the vertical arrows are the canonical isomorphisms, and sign is that from Step 1. We are done. □

8.7 The structure of $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$

Recall that $\Lambda_{G,P}$ is canonically identified with the lattice of characters of the center $Z(\check{Q})$ of the Langlands dual group $\check{Q}$ of Q. For a representation V of $\mathbb{SO}_{2n+1}$ and $\theta \in \Lambda_{G,P}$ write V_θ for the direct summand of V on which $Z(\check{Q})$ acts by θ.

For $\lambda \in \Lambda^+$ write V^λ for the irreducible representation of $\mathbb{SO}_{2n+1}$ of highest weight λ. Write $\omega_i \in \Lambda^+$ for the fundamental coweight of G corresponding to the representation $\wedge^i V^\alpha$ of $\mathbb{SO}_{2n+1}$, $i = 1, \ldots, n$. Let $\mathrm{Loc} : \mathrm{Rep}(\check{Q}) \to \mathrm{Sph}(\mathrm{Gr}_Q)^\natural$ denote the Satake equivalence, normalized to send an irreducible representation of $\check{Q}$ with highest weight μ to $\mathcal{A}_{Q,\mu}$.

Proposition 15. *Let $\lambda \in \Lambda^+$ and θ be the image of λ in $\Lambda_{G,P}$. Then $F^\theta(\mathcal{A}_\lambda) \widetilde{\to} \mathrm{Loc}(V^\lambda_\theta)$ canonically. In particular, $F^\theta(\mathcal{A}_{\omega_i}) \widetilde{\to} \mathcal{A}_{Q,\omega_i}$ for $\langle \theta, \check{\omega}_n \rangle = i$.*

Proof We could similarly define the functor $F^\theta : \mathrm{Sph}(\mathrm{Gr}_G) \to \mathrm{Sph}(\mathrm{Gr}^\theta_Q)$. Write $\mathcal{A}_{\lambda,old}$ for the corresponding object of $\mathrm{Sph}(\mathrm{Gr}_G)$. We claim that $F^\theta(\mathcal{A}_\lambda) \widetilde{\to} F^\theta(\mathcal{A}_{\lambda,old})$ canonically for our particular θ.

Indeed, $S^\theta_P \cap \overline{\mathrm{Gr}}^\lambda_G \hookrightarrow \overline{\mathrm{Gr}}^\lambda_G$ is an open immersion, and the gerbe $\tilde{S}^\theta_P \to S^\theta_P$ is trivial. So, the $*$-restriction of $\mathcal{A}_\lambda$ under $S^\theta_P \cap \overline{\mathrm{Gr}}^\lambda_G \to \widetilde{\mathrm{Gr}}_G$ is the Goresky-MacPherson extension from $S^\theta_P \cap \mathrm{Gr}^\lambda_G$. The assertion follows now from (Proposition 4.3.3 and Theorem 4.3.4,[4]). □

Proposition 16. *i) If $1 \le i \le n$ then $\mathcal{A}_{\omega_i}$ appears in $\mathcal{A}_\alpha^{\otimes i}$.*
ii) For $\lambda, \mu \in \Lambda$ the multiplicity of $\mathcal{A}_{\lambda+\mu}$ in $\mathcal{A}_\lambda \otimes \mathcal{A}_\mu$ is one.

Proof i) Let $\theta \in \Lambda_{G,P}$ be given by $\langle \theta, \check{\omega}_n \rangle = i$. By Proposition 14, $F(\mathcal{A}_\alpha^{\otimes i}) \widetilde{\to} (\mathcal{A}_{Q,\alpha} \oplus \mathcal{A}_{Q,-\alpha})^{\otimes i}$. So, $F^\theta(\mathcal{A}_\alpha^{\otimes i}) \widetilde{\to} \mathcal{A}_{Q,\alpha}^{\otimes i}$. Applying an appropriate symmetrazation functor (either invariants or anti-invariants), one gets a direct summand $\mathcal{V} \subset \mathcal{A}_\alpha^{\otimes i}$ such that $F^\theta(\mathcal{V}) \widetilde{\to} \mathcal{A}_{Q,\omega_i}$.

If $\mathcal{A}_\lambda$ appears in $\mathcal{V}$ then $F^\theta(\mathcal{A}_\lambda) \subset F^\theta(\mathcal{V})$, because F^θ is exact. Besides, $\lambda \le i\alpha$ in the sense that $\mathrm{Gr}^\lambda_G \subset \overline{\mathrm{Gr}}^{i\alpha}_G$, so $\langle \lambda, \check{\omega}_n \rangle \le i$. If $\langle \lambda, \check{\omega}_n \rangle < i$ then $F^\theta(\mathcal{A}_\lambda) = 0$ by Remark 8. If $\langle \lambda, \check{\omega}_n \rangle = i$ then, by Corollary 1, $\mathcal{A}_{Q,\lambda}$ appears in $F^\theta(\mathcal{V}) \widetilde{\to} \mathcal{A}_{Q,\omega_i}$, so $\lambda = \omega_i$. The assertion follows.

ii) Consider the convolution map $m : \overline{\mathrm{Gr}}^\lambda_G \tilde{\times} \overline{\mathrm{Gr}}^\mu_G \to \overline{\mathrm{Gr}}^{\lambda+\mu}_G$ as in Sect. 8.2. Its restriction to the open subscheme $\mathrm{Gr}^\lambda_G \tilde{\times} \mathrm{Gr}^\mu_G \to \mathrm{Gr}^{\lambda+\mu}_G$ is an isomorphism, as follows from ([17], Lemma 4.3 and formula 3.6). We are done. □

Proof of Proposition 14 ii)

Call an object $K \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ *even* (resp., *odd*) if $F^\theta(K) = 0$ unless $\flat(\theta) = 0$ (resp., $\flat(\theta) = 1$). Proposition 11 combined with Proposition 16 shows that $\mathcal{A}_\alpha$ is a tensor generator of $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$. Since $\mathcal{A}_\alpha$ is odd, we get a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ compatible with the tensor structure. Moreover, F' is compatible with the gradings. The uniqueness of the $\mathbb{Z}/2\mathbb{Z}$-grading is clear, because $\mathcal{A}_\alpha$ is irreducible. □

Definition 6. Let $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})^\flat$ be the category of even objects in $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x}) \otimes \mathrm{Vect}^\epsilon$.

By Proposition 14, we get a tensor functor $F' : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_{G,x})^\flat \to \mathrm{Sph}'(\mathrm{Gr}_{Q,x})$. Denote by $F^\natural$ the composition

$$\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat \xrightarrow{F'} \mathrm{Sph}'(\mathrm{Gr}_Q) \widetilde{\to} \mathrm{Sph}(\mathrm{Gr}_Q)^\natural$$

Let $\tilde{h} : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat \to \mathrm{Vect}$ denote the tensor functor $\tilde{h} = h \circ F^\natural$.

Corollary 2. *There is an affine group scheme $\check{G}$ over $\bar{\mathbb{Q}}_\ell$ such that $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$ and the category $\mathrm{Rep}(\check{G})$ of $\bar{\mathbb{Q}}_\ell$-representations of $\check{G}$ are canonically equivalent as tensor categories.*

Proof By Corollary 1, for each nonzero $\lambda \in \Lambda^+$ the rank of $\tilde{h}(\mathcal{A}_\lambda)$ is at least 2. By (Proposition 1.20, [9]), $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$ is a rigid abelian tensor category (cf. Definition 1.7, *loc.cit*) and $\tilde{h} : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat \to \mathrm{Vect}$ is a fibre functor. Our assertion follows now from (Theorem 2.11, *loc.cit.*). $\square$

Write W^λ for the representation of $\check{G}$ corresponding to $\mathcal{A}_\lambda$, $\lambda \in \Lambda^+$. The functor $F^\natural : \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat \to \mathrm{Sph}(\mathrm{Gr}_Q)^\natural$ yields a morphism $\check{Q} \to \check{G}$. By Proposition 13, $W^\alpha = U^\alpha \oplus U^{\alpha *}$, where U^α is the irreducible representation of $\check{Q}$ of highest weight α. Since W^α is a faithful representation of $\check{Q}$, it follows that $\check{Q} \to \check{G}$ is an injection.

Since W^α is a tensor generator of $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$, $\check{G}$ is of finite type. We also get that $\check{G} \subset \mathrm{SL}(W^\alpha)$. Indeed, the only object of rank one in $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$ is $\mathcal{A}_0$, so $\check{G}$ acts trivially on $\det W^\alpha$.

Let $\mathcal{S} \in \mathrm{Rep}(\check{G})$ be such that the strictly full subcategory of $\mathrm{Rep}(\check{G})$, whose objects are isomorphic to subobjects of $\oplus_{i=1}^m \mathcal{S}$, is stable under the tensor structure. Then $\check{Q}$ acts trivially on $F^\natural(\mathcal{S})$, because $\check{Q}$ is connected. If $\check{Q}$ acts trivially on some $F^\natural(\mathcal{A}_\lambda)$ then $\lambda = 0$ by Proposition 15. So, $\mathcal{S}$ is a multiple of $\mathcal{A}_0$. By ([9], 2.22), this implies that $\check{G}$ is connected. Now by (*loc.cit.*, 2.23), $\check{G}$ is reductive.

The above $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$ gives rise to a group homomorphism $\mu_2 \to \check{G}$.

Lemma 13. *For $i = 1, \dots, n$ the multiplicity of W^{ω_i} in $\wedge^i W^\alpha$ is one. If W^λ appears in $\wedge^i W^\alpha$ and $\lambda \neq \omega_i$ then $\langle \lambda, \check{\omega}_n \rangle < i$.*

Proof Let $\theta \in \Lambda_{G,P}$ be given by $\langle \theta, \check{\omega}_n \rangle = i$. The direct summand of $\wedge^i W^\alpha = \wedge^i (U^\alpha \oplus U^{\alpha *})$, on which $Z(\check{Q})$ acts by θ is $\wedge^i U^\alpha$. It follows that $F^\theta(\wedge^i \mathcal{A}_\alpha) = \mathcal{A}_{Q,\omega_i}$, where we denoted by $\wedge^i \mathcal{A}_\alpha$ the object of $\mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)^\flat$ corresponding to $\wedge^i W^\alpha$.

If W^λ appears in $\wedge^i W^\alpha$ then $F^\theta(\mathcal{A}_\lambda) \subset F^\theta(\wedge^i \mathcal{A}_\alpha)$, because F^θ is exact. Besides, $\lambda \le i\alpha$ in the sense that $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}}_G^{i\alpha}$, so $\langle \lambda, \check{\omega}_n \rangle \le i$. If $\langle \lambda, \check{\omega}_n \rangle < i$ then $F^\theta(\mathcal{A}_\lambda) = 0$ by Remark 8. If $\langle \lambda, \check{\omega}_n \rangle = i$ then, by Corollary 1, $\mathcal{A}_{Q,\lambda}$ appears in $F^\theta(\wedge^i \mathcal{A}_\alpha) = \mathcal{A}_{Q,\omega_i}$, so $\lambda = \omega_i$. The assertion follows. $\square$

Proof of Theorem 3

Step 1. Let us show that $\mathcal{A}_\alpha * \mathcal{A}_\alpha \widetilde{\to} \mathcal{A}_{2\alpha} \oplus \mathcal{A}_{\omega_2} \oplus \mathcal{A}_0$ for $n \ge 2$ and $\mathcal{A}_\alpha * \mathcal{A}_\alpha \widetilde{\to} \mathcal{A}_{2\alpha} \oplus \mathcal{A}_0$ for $n = 1$. Indeed, by Proposition 16, $\mathcal{A}_{2\alpha} \oplus \mathcal{A}_{\omega_2}$ appears in $\mathcal{A}_\alpha * \mathcal{A}_\alpha$. Let $\theta \in \Lambda_{G,P}$ be given by $\langle \theta, \check{\omega}_n \rangle = 2$. By Proposition 15, $F^\theta(\mathcal{A}_{2\alpha}) \widetilde{\to} \mathcal{A}_{Q,2\alpha}$ and $F^\theta(\mathcal{A}_{\omega_2}) \widetilde{\to} \mathcal{A}_{Q,\omega_2}$. We have

$$F^\theta(\mathcal{A}_\alpha * \mathcal{A}_\alpha) \widetilde{\to} \mathrm{Loc}((W^\alpha \otimes W^\alpha)_\theta) \widetilde{\to} \mathrm{Loc}(U^\alpha \otimes U^\alpha) \widetilde{\to} \mathcal{A}_{Q,2\alpha} \oplus \mathcal{A}_{Q,\omega_2}$$

So, $\mathcal{A}_\alpha * \mathcal{A}_\alpha \widetilde{\to} \mathcal{A}_{2\alpha} \oplus \mathcal{A}_{\omega_2} \oplus K$ for some $K \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ such that $F^{\theta'}(K) = 0$ unless $\langle \theta', \check{\omega}_n \rangle < 2$. Since $\mathcal{A}_\alpha$ is odd, $\mathcal{A}_\alpha * \mathcal{A}_\alpha$ is even, so K is multiple of $\mathcal{A}_0$. The disired assertion follows now from $\mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_\alpha * \mathcal{A}_\alpha) \widetilde{\to} \mathrm{Hom}(\mathcal{A}_\alpha, \mathcal{A}_\alpha) \widetilde{\to} \bar{\mathbb{Q}}_\ell$.

Step 2. Let us show that $\mathcal{A}_0$ appears in $\wedge^2 \mathcal{A}_\alpha$. Assume the contrary, that is, $\mathcal{A}_0$ appears in $\mathrm{Sym}^2 \mathcal{A}_\alpha$. Then $n \geq 2$ and $\check{G} \subset \mathbb{SO}(W^\alpha)$ for the symmetric form $\mathrm{Sym}^2 W^\alpha \to U^\alpha \otimes U^{\alpha *} \to \bar{\mathbb{Q}}_\ell$.

Let $\check{U}$ (resp., $\check{U}^-$) denote the unipotent radical of the Siegel parabolic $\check{P} \subset \mathbb{SO}(W^\alpha)$ (resp., $\check{P}^- \subset \mathbb{SO}(W^\alpha)$) preserving the isotropic subspace $U^\alpha \subset W^\alpha$ (resp., $U^{\alpha *} \subset W^\alpha$). The Lie algebra $\mathrm{Lie}\, \check{G}$ is a $\check{Q}$-subrepresentation of

$$\mathfrak{so}(W^\alpha) = \mathfrak{gl}(U^\alpha) \oplus \mathrm{Lie}(\check{U}) \oplus \mathrm{Lie}(\check{U}^-)$$

Since $\mathrm{Lie}\, \check{U}$ and $\mathrm{Lie}\, \check{U}^-$ are irreducible $\check{Q}$-modules, $\check{G}$ coincides with one of the groups $\check{Q}, \check{P}, \check{P}^-, \mathbb{SO}(W^\alpha)$. Since $\check{G}$ is reductive, it is either $\check{Q}$ or $\mathbb{SO}(W^\alpha)$. Since W^α is not irreducible as a representation of $\check{Q}$, $\check{G} \neq \check{Q}$, hence $\check{G} = \mathbb{SO}(W^\alpha)$.

Now Lemma 13 shows that $\wedge^n W^\alpha \widetilde{\to} W^{\omega_n} \oplus W^\lambda$ for some $\lambda \in \Lambda^+$ with $\langle \lambda, \check{\omega}_n \rangle < n$. Let $\tilde{U}$ denote the kernel of the contraction map $\wedge^{n-1} U^\alpha \otimes U^{\alpha *} \to \wedge^{n-2} U^\alpha$, this is an irreducible $\check{Q}$-module. By the representation theory for $\mathbb{SO}_{2n}$, we have

- $\tilde{U} \subset W^\lambda \subset \wedge^n (U^\alpha \oplus U^{\alpha *})$ as $\check{Q}$-modules;
- if a weight θ of $Z(\check{Q})$ appears in W^λ then $\langle \theta, \check{\omega}_n \rangle \leq n - 2$;
- for $\langle \theta, \check{\omega}_n \rangle = n - 2$ the direct summand of W^λ on which $Z(\check{Q})$ acts by θ is $\tilde{U}$.

Let θ be the image of λ in $\Lambda_{G,P}$, we get $F^\theta(\mathcal{A}_\lambda) \widetilde{\to} \tilde{U}$. By Corollary 1, $\mathcal{A}_{Q,\lambda} \widetilde{\to} \tilde{U}$. However, the highest weight of $\tilde{U}$ does not lie in Λ_+. This contradiction yields our statement.

Step 3. We know already that $\check{G} \subset \mathbb{Sp}(W^\alpha)$ for the form $\wedge^2 W^\alpha \to U^\alpha \otimes U^{\alpha *} \to \bar{\mathbb{Q}}_\ell$. Let $\check{P} \subset \mathbb{Sp}(W^\alpha)$ (resp., $\check{P}^- \subset \mathbb{Sp}(W^\alpha)$) denote the Siegel parabolic preserving the lagrangian subspace $U^\alpha \subset W^\alpha$ (resp., $U^{\alpha *} \subset W^\alpha$). As in Step 2, one shows that $\check{G}$ coincides with one of the groups $\check{Q}, \check{P}, \check{P}^-, \mathbb{Sp}(W^\alpha)$. Since $\check{G}$ is reductive, it is either $\check{Q}$ or $\mathbb{Sp}(W^\alpha)$. The $\check{Q}$-representation W^α is not irreducible, so $\check{G} = \mathbb{Sp}(W^\alpha)$. $\square$

9. Hecke operators

9.1 According to A.3, inside of $\mathrm{D}(\widetilde{\mathrm{Bun}}_G)$ we have the full triangulated subcategories $\mathrm{D}_\pm(\widetilde{\mathrm{Bun}}_G)$. Let us define for each $K \in \mathrm{Sph}(\widetilde{\mathrm{Gr}}_G)$ a Hecke operator $\mathrm{H}(K, .) : \mathrm{D}(\widetilde{\mathrm{Bun}}_G) \to \mathrm{D}(X \times \widetilde{\mathrm{Bun}}_G)$ sending $\mathrm{D}_\pm(\widetilde{\mathrm{Bun}}_G)$ to $\mathrm{D}_\pm(X \times \widetilde{\mathrm{Bun}}_G)$.

Denote by $\mathcal{H}_G$ the Hecke stack classifying $(\mathcal{F}_G, \mathcal{F}'_G, x \in X, \beta)$, where $\mathcal{F}_G, \mathcal{F}'_G$ are G-torsors on X, and $\beta : \mathcal{F}_G \widetilde{\to} \mathcal{F}'_G \mid_{X - x}$ is an isomorphism. We have the diagram

$$\mathrm{Bun}_G \xleftarrow{p} \mathcal{H}_G \xrightarrow{p'} \mathrm{Bun}_G,$$

where p (resp., p') sends the above point to $\mathcal{F}_G$ (resp., to $\mathcal{F}'_G$). Let $\tilde{\mathcal{H}}_G$ be the stack obtained from $\widetilde{\operatorname{Bun}}_G \times \widetilde{\operatorname{Bun}}_G$ by the base change $\mathcal{H}_G \stackrel{p,p'}{\to} \operatorname{Bun}_G \times \operatorname{Bun}_G$. Denote by $\tilde{p}, \tilde{p}'$ the projections that fit into the diagram

$$\begin{array}{ccccc} \widetilde{\operatorname{Bun}}_G & \stackrel{\tilde{p}}{\leftarrow} & \tilde{\mathcal{H}}_G & \stackrel{\tilde{p}'}{\to} & \widetilde{\operatorname{Bun}}_G \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Bun}_G & \stackrel{p}{\leftarrow} & \mathcal{H}_G & \stackrel{p'}{\to} & \operatorname{Bun}_G \end{array}$$

Recall that the 'trivial' G-torsor $\mathcal{F}^0_G$ on X is given by $M_0 = \mathcal{O}^n_X \oplus \Omega^n$. Write $\operatorname{Bun}_{G,X}$ for the stack classifying triples $(\mathcal{F}_G, x \in X, \nu)$, where $\mathcal{F}_G \in \operatorname{Bun}_G$ and $\nu : \mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{D_x}$ is a trivialization over the formal disk D_x at $x \in X$. Then $\operatorname{Bun}_{G,X}$ is a G_X-torsor over $X \times \operatorname{Bun}_G$. Set $\widetilde{\operatorname{Bun}}_{G,X} = \widetilde{\operatorname{Bun}}_G \times_{\operatorname{Bun}_G} \operatorname{Bun}_{G,X}$.

Denote by γ (resp., γ') the isomorphism $\operatorname{Bun}_{G,X} \times_{G_X} \operatorname{Gr}_{G,X} \widetilde{\to} \mathcal{H}_G$ such that the projection to the first term corresponds to p (resp., to p'). Recall the line bundle $\mathcal{A}$ on Bun_G (cf. 3.2). We have canonically

$$\gamma'^* p^* \mathcal{A} \widetilde{\to} \mathcal{A} \tilde{\boxtimes} \mathcal{L}^{-1}$$

This yields a G_X-torsor $\widetilde{\operatorname{Bun}}_{G,X} \times \widetilde{\operatorname{Gr}}_{G,X} \to \tilde{\mathcal{H}}_G$ extending the G_X-torsor

$$\operatorname{Bun}_{G,X} \times \operatorname{Gr}_{G,X} \to \operatorname{Bun}_{G,X} \times_{G_X} \operatorname{Gr}_{G,X} \stackrel{\gamma'}{\to} \mathcal{H}_G$$

So, for $\mathcal{S} \in \operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})$ and $\mathcal{T} \in \operatorname{D}(\widetilde{\operatorname{Bun}}_G)$ we can form their twisted tensor product $\mathcal{T} \tilde{\boxtimes} \mathcal{S} \in \operatorname{D}(\tilde{\mathcal{H}}_G)$. Set

$$\operatorname{H}(\mathcal{S}, \mathcal{T}) = (\operatorname{supp} \times \tilde{p})_!(\mathcal{T} \tilde{\boxtimes} \mathcal{S}),$$

where $\operatorname{supp} : \tilde{\mathcal{H}}_G \to X$ is the projection. In a similar way, for any $\mathcal{S} \in \operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X^d})$ one defines the functor $\operatorname{H}(\mathcal{S}, .) : \operatorname{D}(\widetilde{\operatorname{Bun}}_G) \to \operatorname{D}(X^d \times \widetilde{\operatorname{Bun}}_G)$.

Recall the functor glob : $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G) \to \operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})$ (cf. 8.3.1). For $K \in \operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$ set $\operatorname{H}(K, \mathcal{T}) = \operatorname{H}(\operatorname{glob}(K), \mathcal{T})$.

The Hecke functors commute with Verdier duality $\mathbb{D}\operatorname{H}(K, \mathcal{T}) \widetilde{\to} \operatorname{H}(\mathbb{D}K, \mathbb{D}\mathcal{T})$, because Gr_G is ind-proper. Besides, they are compatible with the convolution product on $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)$, namely, for $\mathcal{S}_1, \mathcal{S}_2 \in \operatorname{Sph}(\widetilde{\operatorname{Gr}}_{G,X})$ we have canonically $\operatorname{H}(\mathcal{S}_2, \operatorname{H}(\mathcal{S}_1, \mathcal{T})) \widetilde{\to} \operatorname{H}(\mathcal{S}_1 *_X \mathcal{S}_2, \mathcal{T})$.

The *geometric Langlands program for the metaplectic group* would be a trial to understand the action of $\operatorname{Sph}(\widetilde{\operatorname{Gr}}_G)^\natural$ on $\operatorname{D}_-(\widetilde{\operatorname{Bun}}_G)$, that is, to look for automorphic sheaves or, more generally, for a 'spectral decomposition' of $\operatorname{D}_-(\widetilde{\operatorname{Bun}}_G)$ under this action.

Recall that the metaplectic representation is automorphic. In the geometric setting this is reflected in the following Hecke property of Aut. Set

$$\operatorname{St} = \bar{\mathbb{Q}}_\ell[2n-1](\frac{2n-1}{2}) \oplus \bar{\mathbb{Q}}_\ell[2n-3](\frac{2n-3}{3}) \oplus \ldots \oplus \bar{\mathbb{Q}}_\ell[1-2n](\frac{1-2n}{2}),$$

so St has cohomologies in odd degrees only and $\mathbb{D}(\operatorname{St}) \widetilde{\to} \operatorname{St}$ as a complex over $\operatorname{Spec} k$.

Theorem 4. *Over $X \times \widetilde{\mathrm{Bun}}_G$ we have*

$$\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_g) \widetilde{\to} \mathrm{St}[1](\frac{1}{2}) \boxtimes \mathrm{Aut}_s$$
$$\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_s) \widetilde{\to} \mathrm{St}[1](\frac{1}{2}) \boxtimes \mathrm{Aut}_g$$

9.2 *Proof of Theorem 4.*

Let $\mathcal{H}_G^\alpha \subset \mathcal{H}_G$ be the locally closed substack given by the condition that $\mathcal{F}_G$ is in the position α with respect to $\mathcal{F}'_G$ (or, equivalently, $\mathcal{F}'_G$ is in the position α with respect to $\mathcal{F}_G$). Set $\tilde{\mathcal{H}}_G^\alpha = \mathcal{H}_G^\alpha \times_{\mathcal{H}_G} \tilde{\mathcal{H}}_G$.

Lemma 14. *There exist isomorphisms*

$$\kappa, \kappa' : \tilde{\mathcal{H}}_G^\alpha \widetilde{\to} (\widetilde{\mathrm{Bun}}_G \times_{\mathrm{Bun}_G} \mathcal{H}_G^\alpha) \times B(\mu_2),$$

where we used $p : \mathcal{H}_G^\alpha \to \mathrm{Bun}_G$ (resp., $p' : \mathcal{H}_G^\alpha \to \mathrm{Bun}_G$) in the fibred product, and the projection to the first term corresponds to $\tilde{p} : \tilde{\mathcal{H}}_G^\alpha \to \widetilde{\mathrm{Bun}}_G$ (resp., to $\tilde{p}' : \tilde{\mathcal{H}}_G^\alpha \to \widetilde{\mathrm{Bun}}_G$).

Proof A point of $\tilde{\mathcal{H}}_G^\alpha$ is given by $(\mathcal{F}_G, \mathcal{F}'_G, x \in X, \beta) \in \mathcal{H}_G^\alpha$, two 1-dimensional vector spaces $\mathcal{B}, \mathcal{B}'$ with $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$, $\mathcal{B}'^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M')$. Here M, M' are vector bundles on X obtained from $\mathcal{F}_G, \mathcal{F}'_G$ via the standard representation of G.

The symplectic form on M induces a perfect pairing $(M + M')/M \otimes (M + M')/M' \to \Omega(x)/\Omega \widetilde{\to} k$ between these 1-dimensional spaces. Further,

$$\frac{\det \mathrm{R}\Gamma(X, M)}{\det \mathrm{R}\Gamma(X, M')} \widetilde{\to} \frac{(M + M')/M'}{(M + M')/M} \widetilde{\to} ((M + M')/M')^{\otimes 2}$$

Instead of providing $\mathcal{B}, \mathcal{B}'$ we may provide $\mathcal{B}, \mathcal{B}_0$, where $\dim \mathcal{B}_0 = 1$, with an isomorphism $\mathcal{B}_0^2 \widetilde{\to} k$, letting $\mathcal{B}' = \mathcal{B} \otimes ((M + M')/M')^* \otimes \mathcal{B}_0$. This defines κ. The datum of $\mathcal{B}', \mathcal{B}_0$ defines κ'. $\square$

As above, let W denote the nontrivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\mathrm{Spec}\, k \to B(\mu_2)$. For the diagram

$$X \times \widetilde{\mathrm{Bun}}_G \stackrel{\mathrm{supp} \times \tilde{p}}{\leftarrow} \tilde{\mathcal{H}}_G^\alpha \stackrel{\tilde{p}'}{\to} \widetilde{\mathrm{Bun}}_G$$

the Hecke operator writes $\mathrm{H}(\mathcal{A}_\alpha, K) \widetilde{\to} (\mathrm{supp} \times \tilde{p})_!(\tilde{p}'^* K \otimes \kappa^* W)[2n+1](\frac{2n+1}{2})$.

9.2.1 *Stratifications*

Let (x, M) be a k-point of $X \times {}_i\mathrm{Bun}_G$. Denote by Y the fibre of $\mathrm{supp} \times p : \mathcal{H}_G^\alpha \to X \times \mathrm{Bun}_G$ over (x, M). So, Y can be identified with the variety $\bar{Z} - A$ of Sect. 8.5. Let Y_k denote the preimage of ${}_k\mathrm{Bun}_G$ under $Y \hookrightarrow \mathcal{H}_G^\alpha \stackrel{p'}{\to} \mathrm{Bun}_G$. We are going to describe the stratification of Y by the subschemes Y_k.

Recall that $M \in \mathrm{Bun}_{2n}$ with symplectic form $\wedge^2 M \to \Omega$ and $\dim \mathrm{H}^0(M) = i$ (for brevity, in this subsection we omit the argument X in the cohomology groups). For a k-point M' of Y we get

$$\begin{array}{ccccc} & M & \subset & M+M' & \subset M(x) \\ & \cup & & \cup & \\ M(-x) \subset & M \cap M' & \subset & M' & \end{array}$$

and $\dim(M+M')/M = 1$, $\dim(M \cap M')/M(-x)) = 2n-1$. Actually, $(M \cap M')/M(-x)$ is the orthogonal complement to $(M+M')/M$ for the perfect pairing

$$M(x)/M \otimes M/M(-x) \to \Omega(x)/\Omega \widetilde{\to} k$$

induced by the form on M. Let $\pi : Y \to V = \mathbb{P}(M(x)/M)$ be the map sending M' to the line $M+M'/M$. Let N be the image of $\mathrm{H}^0(M) \to M/M(-x)$. Set $j = \dim N$, so $\dim \mathrm{H}^0(M(-x)) = i - j$. Since $M \widetilde{\to} M^* \otimes \Omega$,

$$\mathrm{H}^0(M(-x)) \widetilde{\to} \mathrm{H}^1(M(x))^* \quad \text{and} \quad \mathrm{H}^1(M(-x)) \widetilde{\to} \mathrm{H}^0(M(x))^*$$

The long exact sequence

$$0 \to \mathrm{H}^0(M) \to \mathrm{H}^0(M(x)) \to M(x)/M \to \mathrm{H}^1(M) \to \mathrm{H}^1(M(x)) \to 0$$

shows that $\dim \mathrm{H}^0(M(x)) = i + 2n - j$, because $\dim \mathrm{H}^1(M(x)) = i - j$. We have

$$\mathrm{H}^0(M \cap M') \widetilde{\to} \mathrm{H}^1(M+M')^* \quad \text{and} \quad \mathrm{H}^1(M \cap M') \widetilde{\to} \mathrm{H}^0(M+M')^*,$$

because $(M+M')^* \otimes \Omega \widetilde{\to} M \cap M'$. Note that $\chi(M \cap M') = -1$ and $\chi(M+M') = 1$.

We distinguish three cases

0) $j = 0$. So, $\mathrm{H}^0(M(-x)) = \mathrm{H}^0(M)$ is i-dimensional and $\dim \mathrm{H}^0(M(x)) = 2n$. Then $\mathrm{H}^0(M(-x)) \widetilde{\to} \mathrm{H}^0(M \cap M')$ is of dimension i, and $\dim \mathrm{H}^0(M+M') = i+1$. Clearly, for $M+M' \in \mathbb{P}(M(x)/M)$ fixed we get a 1-dimensional subspace in $(M+M')/(M \cap M')$ generated by $\mathrm{H}^0(M+M')$. So, for $M+M' \in V$ fixed there is a unique M' with $\dim \mathrm{H}^0(M') = i+1$ and for the other M' we have $\dim \mathrm{H}^0(M') = i$.

   Thus, $\pi : Y \to V$ has a section $V \to Y$, which is the closed stratum Y_{i+1}. Its complement is the open stratum Y_i.

1) $0 < j < 2n$. View V as the space of hyperplanes in $M/M(-x)$. We get a nontrivial subspace $V' \subset V$ of hyperplanes that contain N. Distinguish two cases:

   CASE 1a) $N \subset (M \cap M')/M(-x)$ then $\mathrm{H}^0(M \cap M') = \mathrm{H}^0(M)$ is of dimension i, so $\dim \mathrm{H}^0(M+M') = i+1$. In the fibre of $\pi : Y \to V$ over $M+M'/M$ we get a distinguished point corresponding to the subspace of $(M+M')/(M \cap M')$ generated by $\mathrm{H}^0(M+M')$. This point lies in ${}_{i+1}\mathrm{Bun}_G$, and the complement lies in ${}_i\mathrm{Bun}_G$.

   CASE 1b) $N \not\subseteq (M \cap M')/M(-x)$. Then $N \cap (M \cap M')$ is of dimension $j-1$. So, $\dim \mathrm{H}^0(M \cap M') = i-1$ and $\dim \mathrm{H}^0(M+M') = i$. Since $M' \neq M$, we get $M' \in {}_{i-1}\mathrm{Bun}_G$.

So, Y has three nonempty strata in case 1). The map $\pi : \pi^{-1}(V') \to V'$ has a section, which is the closed stratum $Y_{i+1} \widetilde{\to} V'$. The complement to this section is the middle stratum $Y_i = \pi^{-1}(V') - V'$, and the open stratum is $Y_{i-1} = \pi^{-1}(V - V')$.

2) $j = 2n$. Then $\mathrm{H}^0(M) = \mathrm{H}^0(M(x))$ is i-dimensional, so $\dim \mathrm{H}^0(M + M') = i$ and $\dim \mathrm{H}^0(M \cap M') = i - 1$. The image of $\mathrm{H}^0(M) \to (M + M')/(M \cap M')$ is 1-dimensional and equals $M/(M \cap M')$. So, $\dim \mathrm{H}^0(M') = i - 1$, because $M' \neq M$.

   In this case $Y = Y_{i-1}$.

Fix in addition a vector space $\mathcal{B}$ together with $\mathcal{B}^2 \widetilde{\to} \det \mathrm{R}\Gamma(X, M)$.

Proposition 17. *Let K denote the fibre of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_g)$ (resp., of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_s)$) at $(x, M, \mathcal{B}) \in X \times {}_i\widetilde{\mathrm{Bun}}_G$. Then $K = 0$ unless i is odd (resp., even). If i is odd (resp., even) then we have noncanonically $K \widetilde{\to} \mathrm{St}[1 + d_G - i]$.*

Proof g) Consider the case where K is the fibre of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_g)$. Assume i even, so only the stratum Y_i of Y contributes to K.

If $j = 0$ then Y_i is a $\mathbb{G}_m$-torsor over V, and the restriction of Aut_g to a fibre of $\pi : Y_i \to V$ is a nontrivial local system of order two, so $K = 0$ in this case. If $j = 2n$ then $K = 0$ because $Y = Y_{i-1}$. If $0 < j < 2n$ then Y_i is a $\mathbb{G}_m$-torsor over V', and the restriction of Aut_g to a fibre of $\pi : Y_i \to V'$ is a nontrivial local system of order two, so $K = 0$.

Now let i be odd, so only the strata Y_{i-1} and Y_{i+1} contribute to K.

If $j = 0$ then the restriction of Aut_g to Y_{i+1} is isomorphic to $\bar{\mathbb{Q}}_\ell[d_G - i - 1]$ by Theorem 1, because $Y_{i+1} \widetilde{\to} \mathbb{P}^{2n-1}$ is simply-connected. Our assertion follows then from

$$\mathrm{St} \widetilde{\to} \mathrm{R}\Gamma(\mathbb{P}^{2n-1}, \bar{\mathbb{Q}}_\ell)[2n - 1](\frac{2n - 1}{2})$$

If $j = 2n$ then the restriction of Aut_g to Y_{i-1} is isomorphic to $\bar{\mathbb{Q}}_\ell[d_G - i + 1]$, because Y_{i-1} is simply-connected. So, $K \widetilde{\to} \mathrm{St}[1 + d_G - i]$. If $0 < j < 2n$ then the restriction of Aut_g to Y_{i+1} identifies with $\bar{\mathbb{Q}}_\ell[d_G - i - 1]$, because $Y_{i+1} \widetilde{\to} V'$ is simply-connected. The contribution of Y_{i+1} to K is

$$\mathrm{R}\Gamma(V', \bar{\mathbb{Q}}_\ell)[d_G - i + 2n]$$

The restriction of Aut_g to Y_{i-1} is $\bar{\mathbb{Q}}_\ell[d_G - i + 1]$, because any rank one local system of order two on $\pi^{-1}(V - V')$ is trivial. So, the contribution of Y_{i-1} to K is $\mathrm{R}\Gamma_c(V - V', \bar{\mathbb{Q}}_\ell)[d_G - i + 2n]$. The distinguished triangle

$$\mathrm{R}\Gamma_c(V - V', \bar{\mathbb{Q}}_\ell)[d_G - i + 2n] \to K \to \mathrm{R}\Gamma(V', \bar{\mathbb{Q}}_\ell)[d_G - i + 2n]$$

yields the desired isomorphism.

s) In the case where K is the fibre of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_s)$, the argument is similar. $\square$

9.2.2 For $k,r \geq 0$ denote by ${}_{k,r}\mathcal{H}^\alpha_G$ the preimage of ${}_k\operatorname{Bun}_G \times_r \operatorname{Bun}_G$ under $p\times p' : \mathcal{H}^\alpha_G \to \operatorname{Bun}_G \times \operatorname{Bun}_G$. Similarly, define the stack ${}_{k,r}\tilde{\mathcal{H}}^\alpha_G$ by the cartesian square

$$\begin{array}{ccc}
{}_{k,r}\tilde{\mathcal{H}}^\alpha_G & \hookrightarrow & \tilde{\mathcal{H}}^\alpha_G \\
\downarrow & & \downarrow {\scriptstyle \tilde{p}\times\tilde{p}'} \\
{}_k\widetilde{\operatorname{Bun}}_G \times_r \widetilde{\operatorname{Bun}}_G & \hookrightarrow & \widetilde{\operatorname{Bun}}_G \times \widetilde{\operatorname{Bun}}_G
\end{array}$$

The two S_2-coverings over ${}_{k,r}\tilde{\mathcal{H}}^\alpha_G$ obtained from ${}_k\rho : \operatorname{Cov}({}_k\widetilde{\operatorname{Bun}}_G) \to {}_k\widetilde{\operatorname{Bun}}_G$ and from ${}_r\rho : \operatorname{Cov}({}_r\widetilde{\operatorname{Bun}}_G) \to {}_r\widetilde{\operatorname{Bun}}_G$ are canonically isomorphic, namely Lemma 14 implies the following.

Lemma 15. *There is a canonical commutative diagram, where both squares are cartesian*

$$\begin{array}{ccccc}
{}_k\operatorname{Bun}_G & \leftarrow & {}_{k,r}\mathcal{H}^\alpha_G \times B(\mu_2) & \rightarrow & {}_r\operatorname{Bun}_G \\
\downarrow {\scriptstyle {}_k\rho} & & \downarrow & & \downarrow {\scriptstyle {}_r\rho} \\
{}_k\widetilde{\operatorname{Bun}}_G & \stackrel{\tilde{p}}{\leftarrow} & {}_{k,r}\tilde{\mathcal{H}}^\alpha_G & \stackrel{\tilde{p}'}{\rightarrow} & {}_r\widetilde{\operatorname{Bun}}_G
\end{array} \qquad \square$$

Let $\mathcal{U} \subset X \times {}_1\operatorname{Bun}_G$ be the open substack given by $\mathrm{H}^0(X, M(-x)) = 0$. As in Lemma 1, one shows that $\mathcal{U}$ is non empty. In general, $\mathcal{U} \neq X \times {}_1\operatorname{Bun}_G$. Let $\tilde{\mathcal{U}}$ be the preimage of $\mathcal{U}$ in $X \times {}_1\widetilde{\operatorname{Bun}}_G$.

Proposition 18. *The first isomorphism of Theorem 4 holds over $\tilde{\mathcal{U}}$, the second holds over $X \times {}_0\widetilde{\operatorname{Bun}}_G$.*

Proof g) Let $Y(\mathcal{U})$ be the preimage of $\mathcal{U}$ under $\operatorname{supp}\times p : \mathcal{H}^\alpha_G \to X \times \operatorname{Bun}_G$. Write $Y_k(\mathcal{U})$ for the preimage of ${}_k\operatorname{Bun}_G$ under $Y(\mathcal{U}) \hookrightarrow \mathcal{H}^\alpha_G \stackrel{p'}{\to} \operatorname{Bun}_G$. Then $Y_0(\mathcal{U}) \to \mathcal{U}$ (resp., $Y_2(\mathcal{U}) \to \mathcal{U}$) is a fibration with fibre isomorphic to $\mathbb{P}^{2n-2}$ (resp., to $\mathbb{A}^{2n}$).

Let $Y_k(\tilde{\mathcal{U}})$ be the preimage of $Y_k(\mathcal{U})$ in $\tilde{\mathcal{H}}^\alpha_G$. For $k = 0, 2$ the restriction of the local system $\tilde{p}'^*({}_k\mathrm{Aut}) \otimes \kappa^* W$ descends under $Y_k(\tilde{\mathcal{U}}) \to \tilde{\mathcal{U}}$ to a local system, which is canonically identified, by Lemma 15, with $\bar{\mathbb{Q}}_\ell \boxtimes {}_1\mathrm{Aut}$.

By Proposition 17, $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_g)$ vanishes over $X \times_0 \widetilde{\operatorname{Bun}}_G$, and we denote by K the restriction of this complex to $\tilde{\mathcal{U}}$. By decomposition theorem, K is a direct sum of (shifted) irreducible perverse sheaves. We get an isomorphism

$$K \widetilde{\to} {}_1\mathrm{Aut}[d_G - 2n + 1](\frac{d_G - 2n + 1}{2}) \oplus {}_1\mathrm{Aut}[d_G + 2n - 1](\frac{d_G + 2n - 1}{2}) \otimes \mathrm{R}\Gamma(\mathbb{P}^{2n-2}, \bar{\mathbb{Q}}_\ell)$$

The first assertion follows.

s) Set $\mathcal{V} = X \times {}_0\operatorname{Bun}_G$. Let K be the restriction of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut}_s)$ to $\tilde{\mathcal{V}} = X \times {}_0\widetilde{\operatorname{Bun}}_G$. Let $Y(\mathcal{V})$ be the preimage of $\mathcal{V}$ under $\operatorname{supp}\times p : \mathcal{H}^\alpha_G \to X \times \operatorname{Bun}_G$. Write $Y_k(\mathcal{V})$ for the preimage of ${}_k\operatorname{Bun}_G$ under $Y(\mathcal{V}) \hookrightarrow \mathcal{H}^\alpha_G \stackrel{p'}{\to} \operatorname{Bun}_G$. Then $Y_1(\mathcal{V}) \to \mathcal{V}$ is a fibration with fibre isomorphic to $\mathbb{P}^{2n-1}$.

Let $Y_1(\tilde{\mathcal{V}})$ be the preimage of $Y_1(\mathcal{V})$ in $\tilde{\mathcal{H}}^\alpha_G$. By Lemma 15, the $*$-restriction of $\tilde{p}'^*({}_1\mathrm{Aut}) \otimes \kappa^* W$ descends under $Y_1(\tilde{\mathcal{V}}) \to \tilde{\mathcal{V}}$ to a local system canonically identified with $\bar{\mathbb{Q}}_\ell \boxtimes {}_0\mathrm{Aut}$. By decomposition theorem, one gets an isomorphism

$$K \widetilde{\to} {}_0\mathrm{Aut} \otimes \mathrm{R}\Gamma(\mathbb{P}^{2n-1}, \bar{\mathbb{Q}}_\ell)[d_G + 2n](\frac{d_G + 2n}{2})$$

We are done. □

By decomposition theorem, $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut})$ is a direct sum of (shifted) irreducible perverse sheaves. Proposition 18 implies that $\mathrm{St}[1](\frac{1}{2}) \boxtimes \mathrm{Aut}$ appears in it as a direct summand. But according to Proposition 17, all the fibres of $\mathrm{H}(\mathcal{A}_\alpha, \mathrm{Aut})$ and of $\mathrm{St}[1](\frac{1}{2}) \boxtimes \mathrm{Aut}$ are isomorphic. This concludes the proof of Theorem 4.

Appendix A.

A.1 For the convenience of the reader we collect here some generalities on group actions.

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of stacks, $G \to \mathcal{Z}$ be a group scheme over $\mathcal{Z}$. Write m_G for the product in G and $1_G : \mathcal{Z} \to G$ for the unit section. Following [5], an action of G on $\mathcal{Y}$ over $\mathcal{Z}$ is the data of a 1-morphism $m : G \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ over $\mathcal{Z}$, a 2-morphism $\mu : m \circ (m_G \times \mathrm{id}) \Rightarrow m \circ (\mathrm{id} \times m)$ making the following diagram 2-commutative

$$\begin{array}{ccc} G \times_{\mathcal{Z}} G \times_{\mathcal{Z}} \mathcal{Y} & \stackrel{m_G \times \mathrm{id}}{\to} & G \times_{\mathcal{Z}} \mathcal{Y} \\ \downarrow \mathrm{id} \times m & & \downarrow m \\ G \times_{\mathcal{Z}} \mathcal{Y} & \stackrel{m}{\to} & \mathcal{Y}, \end{array}$$

and a 2-morphism $\epsilon : m \circ (1_G \times \mathrm{id}_{\mathcal{Y}}) \to \mathrm{id}_{\mathcal{Y}}$. They should satify two axioms: an associativity condition with respect to any 3 objects in G (cf. diagram (6.1.3) in loc.cit.); ϵ is compatible with μ (cf. diagrams (6.1.4) in loc.cit.). The fact that m is a $\mathcal{Z}$-morphism means that the diagram

$$\begin{array}{ccc} G \times_{\mathcal{Z}} \mathcal{Y} & \stackrel{m}{\to} & \mathcal{Y} \\ \downarrow \mathrm{pr}_2 & & \downarrow f \\ \mathcal{Y} & \stackrel{f}{\to} & \mathcal{Z} \end{array}$$

is 2-commutative.

For a line bundle L on $\mathcal{Y}$ we have a notion of *G-equivariant structure* on L (cf. [14], Definition 2.8). A version of this notion for an ℓ-adic complex is as follows.

Definition 7. A *G-equivariant structure* on $K \in \mathrm{D}(\mathcal{Y})$ is an isomorphism $\lambda : m^*K \widetilde{\to} \mathrm{pr}_2^* K$ such that two diagrams commute

$$\begin{array}{ccc} (m_G \times \mathrm{id}_{\mathcal{Y}})^* m^* K & \stackrel{\lambda}{\to} & (m_G \times \mathrm{id}_{\mathcal{Y}})^* \mathrm{pr}_2^* K \\ \downarrow \mu & & \downarrow \lambda \\ (\mathrm{id}_G \times m)^* m^* K & \stackrel{\lambda}{\to} & (\mathrm{id}_G \times m)^* \mathrm{pr}_2^* K = \mathrm{pr}_{23} m^* K \end{array}$$

and

$$\begin{array}{ccc} (1_G \times \mathrm{id}_{\mathcal{Y}})^* m^* K & & \\ \downarrow \lambda & \searrow \epsilon & \\ (1_G \times \mathrm{id}_{\mathcal{Y}})^* \mathrm{pr}_2^* K & = & K, \end{array}$$

where $\mathrm{pr}_2 : G \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$ and $\mathrm{pr}_{23} : G \times_{\mathcal{Z}} G \times_{\mathcal{Z}} \mathcal{Y} \to G \times_{\mathcal{Z}} \mathcal{Y}$ are the projections.

A.2 Let $f : \mathcal{Y} \to \mathcal{Z}$ be a representable morphism of algebraic stacks, $G \to \mathcal{Z}$ be a group scheme over $\mathcal{Z}$ acting on $\mathcal{Y}$ over $\mathcal{Z}$. By definition, $\mathcal{Y}$ is a *G-torsor over $\mathcal{Z}$* if, locally in flat topology of $\mathcal{Z}$, $\mathcal{Y}$ is isomorphic to G over $\mathcal{Z}$ as a G-scheme.

Assume that $\mathcal{Z}$ is locally of finite type. The notion of a perverse sheaf localizes in the smooth topology, so we have a notion of a perverse sheaf on $\mathcal{Z}$. For the same reason, if $G \to \mathcal{Z}$ is of finite type and smooth of relative dimension d then the functor $K \mapsto f^*K[d]$ is an equivalence of the category of perverse sheaves $P(\mathcal{Z})$ on $\mathcal{Z}$ with the category of G-equivariant perverse sheaves $P_G(\mathcal{Y})$ on $\mathcal{Y}$.

A.3 Let $\mathcal{A}$ be a line bundle on a scheme S. Let $\tilde S \to S$ denote the μ_2-gerbe of square roots of $\mathcal{A}$ (cf. 3.3.1). Since μ_2 acts on $\tilde S$ by 2-automorphisms of the identity $\mathrm{id} : \tilde S \to \tilde S$, μ_2 acts on any $K \in \mathrm{D}(\tilde S)$. Write $\pi : \tilde S \to S$ for the structural morphism.

Lemma 16. *1) The functor π^* is an equivalence of the category of perverse sheaves on S with the category of those perverse sheaves on $\tilde S$ on which μ_2 acts trivially.*
2) The functor $\pi^* : D(S) \to D(\tilde S)$ is fully faithful, its image $D_+(S)$ is a full triangulated subcategory of $D(\tilde S)$.
3) For $K \in D(\tilde S)$ the following are equivalent

- *i) $-1 \in \mu_2$ acts as -1 on each cohomology sheaf of K*
- *ii) $\pi_!K = 0$*
- *iii) $\pi_*K = 0$.*

Let $D_-(\tilde S) \subset D(\tilde S)$ be the full triangulated subcategory of objects satisfying these conditions.
4) For any $K_\pm \in D_\pm(\tilde S)$ we have $\operatorname{Hom}_{D(\tilde S)}(K_+, K_-) = 0$ and $\operatorname{Hom}_{D(\tilde S)}(K_-, K_+) = 0$. For $K \in D(\tilde S)$ there exist $K_\pm \in D_\pm(\tilde S)$ such that $K \widetilde{\to} K_+ \oplus K_-$.

Proof 1a) In the case $\mathcal{A} = \mathcal{O}_S$ consider the presentation $i : S \to B(S/\mu_2)$. The functor i^* identifies the category of perverse sheaves on $B(S/\mu_2)$ with the category of perverse sheaves on S equipped with an action of the group $\mu_2(S)$.
1b) In general we have a carthesian square

$$\begin{array}{ccc} \tilde S & \xrightarrow{\pi} & S \\ \uparrow h & & \uparrow \pi \\ \tilde S \times B(\mu_2) & \xrightarrow{\mathrm{pr}} & \tilde S, \end{array}$$

where h sends a T-point $(\mathcal{B}, \mathcal{B}_0,\ \mathcal{B}^2 \widetilde{\to} \mathcal{A}\mid_T\ \ \mathcal{B}_0^2 \widetilde{\to} \mathcal{O}_T)$ to $\mathcal{B} \otimes \mathcal{B}_0$ for any S-scheme T.

If F is a perverse sheaf on $\tilde S$ on which μ_2 acts trivially, then $\mu_2 \times \mu_2$ acts trivially on h^*F. By 1a) we then get an isomorphism $h^*F \widetilde{\to} \mathrm{pr}^* F$ satisfying the usual cocycle condition. So, there is an isomorphism $F \widetilde{\to} \pi^*H$ for some perverse sheaf H on S.

2) The map π is smooth of relative dimension zero, and $\pi_!\bar{\mathbb{Q}}_\ell \widetilde{\to} \bar{\mathbb{Q}}_\ell$. It follows formally that π^* is fully faithful.

3) The functors $\pi_!$ and π_* are exact with respect to the usual t-structure. So, $\pi_! K = 0$ iff $\pi_!(H^i(K)) = 0$ for each i. The latter is equivalent to requiring that -1 acts nontrivially on $H^i(K)$ for each i. Similarly for π_*.
4) Given $K_- \in D_-(\tilde{S})$ and $K_+ \widetilde{\to} \pi^* L \in D_+(\tilde{S})$ we have

$$\mathrm{Hom}(K_-, K_+) \widetilde{\to} \mathrm{Hom}(K_-, \pi^! L) \widetilde{\to} \mathrm{Hom}(\pi_! K_-, L) = 0$$

and

$$\mathrm{Hom}(K_+, K_-) \widetilde{\to} \mathrm{Hom}(\pi^* L; K_-) \widetilde{\to} \mathrm{Hom}(L, \pi_* K_-) = 0$$

We claim that for each $K \in D(\tilde{S})$ the adjointness map $\pi_*\pi^*\pi_* K \to \pi_* K$ is an isomorphism. Since our derived categories are bounded, by devissage we may assume that K is placed in cohomological dimension zero. Then $K \widetilde{\to} K_0 \oplus K_1$, where -1 acts on K_0 (resp., on K_1) as 1 (resp., as -1). Clearly, $\pi^*\pi_* K_0 \widetilde{\to} K_0$ and $\pi_* K_1 = 0$, so $\pi_*\pi^*\pi_* K \widetilde{\to} \pi_* K$.

For $K \in D(\tilde{S})$ let K_- be a cone of the adjointness map $\pi^*\pi_* K \to K$ then $\pi_* K_- = 0$. The triangle $\pi^*\pi_* K \to K \to K_-$ splits, because $\mathrm{Hom}(K_-, \pi^*\pi_* K[1]) = 0$. $\square$

Let G be an algebraic group acting on S, assume that $\mathcal{A}$ is equipped with a G-equivariant structure. Then G acts on $\tilde{S}$, and the projection $\tilde{S} \to S$ is G-equivariant.

The stack $\tilde{S}$ is equipped with the universal line bundle $\mathcal{B}_u$ together with $\mathcal{B}_u^2 \widetilde{\to} \mathcal{A} \mid_{\tilde{S}}$. One checks that $\mathcal{B}_u$ is G-equivariant.

Let G act on the trivial gerbe $S \times B(\mu_2)$ as the product of the action of G on S with the trivial action on $B(\mu_2)$. The following lemma is straightforward.

Lemma 17. *Let $\mathcal{B}$ be a G-equivariant line bundle on S equipped with a G-equivariant isomorphism $\mathcal{B}^2 \widetilde{\to} \mathcal{A}$. Then $\mathcal{B}$ yields a G-equivariant trivialization $\tilde{S} \widetilde{\to} S \times B(\mu_2)$.* $\square$

A.4 Let S be a normal variety with a $\mathbb{G}_m$-action, $\mathcal{A}$ be a $\mathbb{G}_m$-equivariant line bundle on S. Write $\tilde{S} \to S$ for the gerbe of square roots of $\mathcal{A}$. Let $S_0 \subset S$ be the variety of fixed points. For a connected component C of S_0 set

$$S^+(C) = \{s \in S \mid \lim_{t \to 0} ts \in C\} \text{ and}$$

$$S^- = \{s \in S \mid \lim_{t \to \infty} ts \in C\}$$

Let S^+ (resp., S^-) denote the disjoint union of $S^+(C)$ (resp., of $S^-(C)$) indexed by the connected components of S_0. Write $\tilde{S}^+$ (resp., $\tilde{S}^-, \tilde{S}_0$) for the restriction of the gerbe $\tilde{S} \to S$ to the corresponding scheme. Let $f^\pm : \tilde{S}_0 \to \tilde{S}^\pm$ and $g^\pm : \tilde{S}^\pm \to \tilde{S}$ denote the corresponding (representable) maps. Following [3], define *hyperbolic localization* functors $\mathrm{D}(\tilde{S}) \to \mathrm{D}(\tilde{S}_0)$ by

$$K^{!*} = (f^+)^!(g^+)^* K, \qquad K^{*!} = (f^-)^*(g^-)^! K$$

The following generalization of Theorem 1 from *loc.cit.* is straightforward.

Proposition 19. *There is a natural map $i_S : K^{*!} \to K^{!*}$ functorial in $K \in \mathrm{D}(\tilde{S})$. Assume that there is a covering of S by open $\mathbb{G}_m$-invariant subschemes U_i and $\mathbb{G}_m$-invariant trivializations $\xi_i : \mathcal{A}\mid_{U_i} \widetilde{\to} \mathcal{O}\mid_{U_i}$. Then for $\mathbb{G}_m$-equivariant $K \in \mathrm{D}(\tilde{S})$ the map i_S is an isomorphism.*

Proof The map is constructed as in (*loc.cit.*, Sect. 2). Let $\tilde{U}_i$ denote the restriction of $\tilde{S}$ to U_i. It suffices to show the desired map is an isomorphism over $\tilde{U}_i$ for any perverse sheaf $K \in P(\tilde{S})$. The trivialization ξ_i induces $\mathbb{G}_m$-equivariant section $U_i \to \tilde{U}_i$ of the gerbe $\tilde{U}_i \to U_i$. One concludes applying Theorem 1 from *loc.cit.* for $K\mid_{U_i}$. $\square$

Assume in addition that there is a $\mathbb{G}_m$-equivariant section $S^+ \to \tilde{S}^+$ of the gerbe $\tilde{S}^+ \to S^+$. Let $h^+ : S^+ \to S_0$ be the map sending s to $\lim_{t\to 0} ts$. Then for any $\mathbb{G}_m$-equivariant object $K \in \mathrm{D}(\tilde{S})$ we have $K^{!*} \widetilde{\to} (h^+ \times \mathrm{id})_!(g^+)^* K$ canonically. Here $h^+ \times \mathrm{id} : \tilde{S}^+ \widetilde{\to} S^+ \times B(\mu_2) \to S_0 \times B(\mu_2) = \tilde{S}_0$.

Appendix B. Weil representation and the sheaf S_M

B.1 Let $k = \mathbb{F}_q$ be a finite field with q odd. Let M be a symplectic space over k of dimension $2d$. The sheaf S_M introduced in Sect. 4.4 has its origin in the Weil representation, this is what we are going to explain.

Consider the Heisenberg group $H(M) = M \oplus k$ with operation

$$(m, a)(m', a') = (m + m', a + a' + \frac{1}{2}\langle m, m'\rangle)$$

Fix an additive character $\psi : k \to \bar{\mathbb{Q}}_\ell^*$. There exists a unique up to isomorphism irreducible representation of $H(M)$ over $\bar{\mathbb{Q}}_\ell$ with central character ψ. Let (ρ, S_ψ) be such representation. It yields an exact sequence

$$1 \to \bar{\mathbb{Q}}_\ell^* \to \tilde{G} \to G \to 1 \tag{37}$$

with $G = \mathbb{S}\mathrm{p}(M)$. Here

$$\tilde{G} = \{g, M[g]) \mid g \in G, M[g] \in \operatorname{Aut} S_\psi, \quad \rho(gm, a) \circ M[g] = M[g] \circ \rho(m, a)\}$$

Let $\mathcal{L}(M)$ denote the variety of Lagrangian subspaces of M. For $L \in \mathcal{L}(M)$ let $\chi_L : L \oplus k \to \bar{\mathbb{Q}}_\ell^*$ send (l, a) to $\psi(a)$. Set

$$S_{L,\psi} = \operatorname{Ind}_{L\oplus k}^{H(M)} \chi_L = \{f : H(M) \to \bar{\mathbb{Q}}_\ell \mid f(xh) = \chi_L(x) f(h) \text{ for } x \in L \oplus k\}$$

For each $L \in \mathcal{L}(M)$ there is a pair $(v_L \in S_\psi, f_L \in S_\psi^*)$ which is $(L \oplus k, \chi_L)$-inivariant. Normalize it by $f_L(v_L) = 1$, so any such pair is $(a v_L, a^{-1} f_L)$ with $a \in \bar{\mathbb{Q}}_\ell^*$. Specifying such pair is equivalent to specifying an isomorphism of $H(M)$-modules $S_\psi \widetilde{\to} S_{L,\psi}$ such that the image of f_L becomes the evaluation at zero $f_{L,st} \in S_{L,\psi}^*$ (resp., v_L becomes the function $v_{L,st} : H(M) \to \bar{\mathbb{Q}}_\ell$ supported at $L \oplus k$ with $v_{L,st}(0) = 1$).

Let $P_L \subset G$ be the Seigel parabolic subgroup preserving L. Restricting (37) we get an exact sequence

$$1 \to \bar{\mathbb{Q}}_\ell^* \to \tilde{P}_L \to P_L \to 1$$

The action of $\tilde{P}_L$ on $\bar{\mathbb{Q}}_\ell f_L$ yields a character $\tilde{P}_L \to \bar{\mathbb{Q}}_\ell^*$ that splits this sequence (the group $\tilde{P}_L$ acts on $\bar{\mathbb{Q}}_\ell v_L$ by the opposite character).

The *finite-dimensional theta-function* is $\theta_L : P_L \backslash \tilde{G}/P_L \to \bar{\mathbb{Q}}_\ell$ given by $\theta_L(g) = f_L(gv_L)$, it does not depend on the choice of the pair (v_L, f_L).

B.2 Let $L_1, L_2 \in \mathcal{L}(M)$. For $f \in S_{L_1,\psi}$ and $z \in L_2 \oplus k$ the function $f(zh)\chi_{L_2}^{-1}(z)$ depends only on the image of z in L_2, so we may set

$$(F_{L_1,L_2}(f))(h) = \int_{L_2} f(zh)\chi_{L_2}^{-1}(z)dz,$$

where dz is the Haar measure on L_2 such that the volume of a point is one. Then $F_{L_1,L_2} : S_{L_1,\psi} \widetilde{\to} S_{L_2,\psi}$ is an isomorphism of $H(M)$-modules.

One checks that $F_{L_2,L_1} \circ F_{L_1,L_2} \in \operatorname{Aut}(S_{L_1,\psi})$ is the multiplication by $q^{d+\dim(L_1 \cap L_2)}$.

Definition 8. For $L_1, L_2, V \in \mathcal{L}(M)$ with $V \cap L_i = 0$ define $\theta(L_1, L_2, V) \in \bar{\mathbb{Q}}_\ell^*$ by

$$F_{L_2,L_1} \circ F_{V,L_2} \circ F_{L_1,V} = \theta(L_1, L_2, V)$$

We have $L_1 = \{(bu + u) \mid u \in L_2\}$ for uniquely defined $b : L_2 \to V$. The symplectic form on M yields $L_2 \widetilde{\to} V^*$, so b becomes an element of $\operatorname{Sym}^2 V$. From definitions it follows that

$$\theta(L_1, L_2, V) = q^d \int_{V^*} \psi(\frac{1}{2}\langle bv^*, v^*\rangle)dv^*, \tag{38}$$

where dv^* is the Haar measure on V^* such that the volume of a point is one.

Denote by $\tilde{\mathcal{Y}}(k)$ the set of isomorphism classes of collections $L_1, L_2 \in \mathcal{L}(M)$, a one-dimensinal space $\mathcal{B}$ together with $\mathcal{B}^{\otimes 2} \widetilde{\to} (\det L_1) \otimes (\det L_2)$. So, $\tilde{\mathcal{Y}}(k)$ is a two-sheeted covering of the set $\mathcal{Y}(k)$ of G-orbits on $\mathcal{L}(M) \times \mathcal{L}(M)$. Remind that $\mathcal{Y}(k)$ contains $d+1$ element.

Given a triple $L_1, L_2, V \in \mathcal{L}(M)$ with $L_i \cap V = 0$, the form on M yields isomorphisms $L_1 \widetilde{\to} V^* \widetilde{\to} L_2$. So, $(L_1, L_2, \mathcal{B} = \det V^*)$ is a point of $\tilde{\mathcal{Y}}(k)$. Now Proposition 5 implies that $\theta(L_1, L_2, V)$ depends only on the image of (L_1, L_2, V) in $\tilde{\mathcal{Y}}(k)$, so defining a function

$$\theta : \tilde{\mathcal{Y}}(k) \to \bar{\mathbb{Q}}_\ell$$

which is (up to a constant) the trace of Frobenius of the sheaf S_M. It is well-known that for $(L_1, L_2, \mathcal{B}) \in \tilde{\mathcal{Y}}(k)$ with $i = \dim(L_1 \cap L_2)$ one gets

$$\theta(L_1, L_2, \mathcal{B})^2 = \left(\frac{-1}{q}\right)^{d-i} q^{3d+i},$$

where $\left(\frac{-1}{q}\right) = \begin{cases} 1, & \text{if } -1 \in k^2 \\ -1, & \text{otherwise} \end{cases}$

B.3 Remind that we fixed a square root $q^{\frac{1}{2}}$ of q in $\bar{\mathbb{Q}}_\ell$ (cf. 3.1). For $L_1, L_2 \in \mathcal{L}(M)$ set

$$\mathcal{F}_{L_1,L_2} = q^{\frac{1}{2}(-d-\dim(L_1\cap L_2))} F_{L_1,L_2}$$

The following is a version of the Maslov index (cf. [15], appendix to chapter 1).

Definition 9. For $L_1, L_2, L_3 \in \mathcal{L}(M)$ define $\gamma(L_1, L_2, L_3) \in \bar{\mathbb{Q}}_\ell^*$ by

$$\mathcal{F}_{L_2,L_1} \circ \mathcal{F}_{L_3,L_2} \circ \mathcal{F}_{L_1,L_3} = \gamma(L_1, L_2, L_3)$$

Here are its immediate properties (cf. also *loc.cit.*).

Proposition 20. *1)* $\gamma(L_1, L_2, L_3) = \gamma(L_1, L_3, L_2)^{-1} = \gamma(L_2, L_1, L_3)^{-1}$.

2) $\gamma(gL_1, gL_2, gL_3) = \gamma(L_1, L_2, L_3)$ *for* $g \in G$.

3) If $L_1, L_2, L_3, L_4 \in \mathcal{L}(M)$ *then*

$$\gamma(L_1, L_2, L_3)\gamma(L_1, L_4, L_2) = \gamma(L_3, L_4, L_2)\gamma(L_1, L_4, L_3) \qquad \square$$

This implies that the function $(g_1, g_2) \mapsto \gamma(L, g_1 L, g_1 g_2 L)$ is a 2-cocycle of G. This is the cocycle defining the extension (37). In our case of finite field k this extension splits ([18], chapter 2, II.1).

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References

[1] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions, Comm. Math. Phys. 164 (1994), 385-419

[2] A. Beauville, Y. Laszlo, Un lemme de descente, Comptes Rendus Acad. Sci. Paris 320 Série I (1995), 335-340

[3] T. Braden, Hyperbolic localization of intersection cohomology, Transform. Groups 8 (2003), no. 3, 209–216

[4] A. Braverman, D. Gaitsgory, Geometric Eisenstein series, Inv. Math. 150 (2002), 287-384

[5] L. Breen, Bitorseurs et cohomologie non abélienne, The Grothendieck Festschrift vol. 1, Progress in Math. vol. 86 (1990) 401-476

[6] A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, preprint

[7] P. Deligne, Le déterminant de la cohomologie, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 93–177, Contemp. Math., 67, Amer. Math. Soc., Providence, RI (1987)

[8] P. Deligne, letter to D. Kazhdan, 6 March 1985

[9] P. Deligne, J.S.Milne, Tannakian categories, in: Hodge cycles, motives and Shimura varieties, Lecture Notes in Math. 900 (1982)

[10] G. Faltings, Algebraic loop groups and moduli spaces of bundles, J. Eur. Math. Soc. 5, 41-68 (2003)

[11] D. Gaitsgory, Construction of central elements in the affine Hecke algebra via nearby cycles, Invent. Math. 144 (2001), no. 2, 253–280

[12] R. Howe, θ-series and invariant theory, Proc. of Symp. in Pure Math., vol. 33 (1979), part 1, 275-285

[13] G. Laumon, Transformation de Fourier homogène, Bull. Soc. Math. France 131 (2003), no. 4, 527–551

[14] Y. Laszlo, Linearization of group stack actions and the Picard group of the moduli of SL_r/μ_s-bundles on a curve. Bull. Soc. Math. France 125 (1997), no. 4, 529–545.

[15] G. Lion, M. Vergne, The Weil representation, Maslov index and theta series, Progress in Math. 6 (1980)

[16] C. Moore, Group extensions of p-adic and adelic linear groups, Publ. IHES 35 (1968), 5-70

[17] I. Mirković, K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, math.RT/0401222 (to appear in Ann. of Math.)

[18] C. Moeglin, M.-F. Vigneras, J.L. Waldspurger, Correspondence de Howe sur un corps p-adique, Lecture Notes in Math. 1291 (1987)

[19] D. Prasad, Weil Representation, Howe duality, and the Theta correspondence (lectures given in Montreal) http://www.mri.ernet.in∼mathweb/dprasad.html

[20] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. 111 (1964), 143-211.